Pseudotriangulations and Rigidity

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1. Pseudo-Triangulating

Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add edges...
Start with a point set... form the convex hull.
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add one edge
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add two edges
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add three edges
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add four edges
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add five edges
Start with a point set . . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add six edges
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add seven edges
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add eight edges
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add nine edges
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add ten edges
Start with a point set...form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add eleven edges
Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

Add twelve edges - *Pseudo-Triangulation*
2. Properties

Theorem 1 (Streinu - 2000)
The following are equivalent

- $G$ is a pseudo-triangulation with the minimum number of edges.
- $G$ is a pointed pseudo-triangulation
- $G$ is a pseudo-triangulation with exactly $2n - 3$ edges
- $G$ is non-crossing, pointed, and has $2n - 3$ edges
- $G$ is non-crossing, pointed, and maximal with this property
Corollary 1 If any of the above conditions are satisfied, then \( G \) is generically minimally rigid in the plane and any realization of \( G \) as a pseudo-triangulation is 1’st order rigid.

Theorem 2 Every planar graph which is generically minimally rigid has a realization as a pointed pseudo-triangulation.

Proof 1 uses an inductive construction together with topological information.

Proof 2 uses linear algebra - Tutte’s approach to drawing a graph.
3. Definition of CPPT

A \textit{combinatorial pointed pseudo-triangulation} (cppt) is an assignment of labels, \textit{big} and \textit{small}, to the angles of a plane graph such that

- every vertex has exactly one big angle,
- every interior face as exactly three small angles
- the outside face has only big angles.

\( G \) has

\[-n \text{ vertices,} \]
\[-e \text{ edges and} \]
\[-f \text{ faces.}\]

Necessary condition for the existence of a cppt:

\[ e = 2n - 3 \]

\text{(Since } n - e + f = 2 \text{ and } 3f - 3 + n = 2e.\text{)}
4. Combinatorial CPPT

A graph in the plane
A combinatorial pseudo-triangulation
A topological realization
5. Examples

A Combinatorial Pseudotriangulation
Orient edges away from the pointed vertex
Delete all non-oriented edges
Triangulate the pseudotriangles
Start again
Add combinatorial angles

\[
\begin{aligned}
& a \\
& b \\
& c \\
& d \\
& e \\
& f \\
& g \\
& a \\
& \quad G
\end{aligned}
\]
Orient away from the large angles.
Form $G^*$
convexify
Back to $G$. 
6. Directed Tutte method

**Theorem 3** From every interior vertex of $G^*$ there are three vertex disjoint paths to the boundary, consequently $G^*$ can be drawn with straight non-crossing lines in the plane in such a way that a given positive stress on all directed edges is resolved.
7. Schnyder trees.
Pseudo-Triangulating Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

\( G_1 \)

\( a (2,2) \)

\( b (3,1) \)

\( c (8,2) \)

\( d (8,5) \)

\( e (2,8) \)

\( f (1,8) \)

\( g (5,5) \)
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Some Examples:

\[ G^*_2 \]
8. A bad example
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10. Reciprocal Figures

We want to draw the geometric dual using the same edge directions.

Construction

Use a framework with a resolvable stress, non-zero on every edge, for example a cycle in the rigidity matroid. Such a cycle corresponds to a pseudo-triangulation with one non-pointed vertex.
Theorem 4 If a generic 2-cycle is realized as a pseudo-triangulation, then the reciprocal diagram is also a pseudo-triangulation.
Lemma 1 There is, up to rotation, a one to one correspondence between the set of pseudo-triangles $T$ and the set of PTC cells, $C(T)$, such that the vector paths between the distinguished vertices on the boundary of $C(T)$ are translations and half-turns of the pseudo-arcs of $T$. 
proof
Lemma 2 Given a framework, together with a resolvable stress \( s \) and a pseudo-triangle \( T \). Suppose that \( T \) is a face in the rotation system which governs the reciprocal. The following are equivalent:

1. The cyclic ordering around the reciprocal vertex is the reverse of the cyclic ordering around \( T \).

2. The reciprocal figure is pointed at the vertex corresponding to \( T \).

3. There is exactly one improper sign change on the stresses as one reads around \( T \).
11. Some Examples:
Seven Wheel 1:
Seven Wheel 2:
Center segment revolves:

Graph not in a plane embedding.
Center segment revolves and rotates:

Graph not in a plane embedding.
A non-planar reciprocal: