Applying Burnside's lemma to a one-dimensional Escher problem

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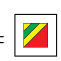
Our point of departure is the paper [7] in which a problem of M. C. Escher is solved using methods of contemporary combinatorics, in particular, Burnside's lemma. Escher originally determined (by laborously examining multitudes of sketches) how many different patterns would result by repeatedly translating a 2×2 square having its four unit squares filled with copies of an asymmetric motif in any of four aspects. In this note we simplify the problem from two dimensions to one dimension but at the same time we generalize it from the case in which a 2×2 block stamps out a repeating planar pattern to the case in which a $1 \times n$ block stamps out a repeating strip pattern.

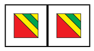
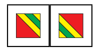
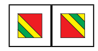
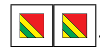
The 1×2 case

Suppose we are tiling a strip by a single rectangle containing an asymmetric motif, say , a motif taken from South African beadwork which is a rectangle divided by a diagonal into two triangles, one solid red, and the other yellow with a green stripe. The original motif has three additional aspects, namely the motif rotated by 180°, reflected in a vertical line and in a horizontal line. We note the motif by b and its other aspects as follows:

$\mathsf{b} =$  $\mathsf{p} =$  $\mathsf{q} =$  $\mathsf{d} =$ 

since the letters p, q and d are the corresponding aspects of the letter b under these transformations. This notation was first introduced in [9] to encode the symmetry groups of strip patterns.

Assume that we may select any two aspects from $\{\mathsf{b}, \mathsf{q}\}$ (with repetition allowed) to form a signature for a 1×2 block of two rectangles containing those aspects of the motif. There are four possible signatures:

bb :  bq :  qb :  qq : .

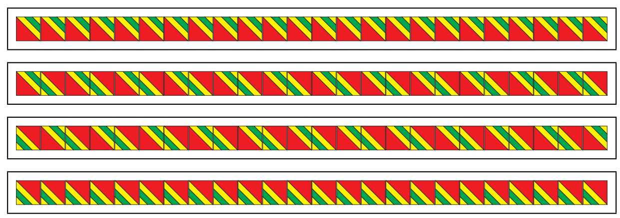
By repeating a 1×2 block horizontally and removing the outline of the rectangles, each signature determines uniquely a 2–way infinite strip pattern:

$\mathsf{bb}^* = \cdots \mathsf{bbbbbb} \cdots =$

$\mathsf{bq}^* = \cdots \mathsf{bqbqbq} \cdots =$

$\mathsf{qb}^* = \cdots \mathsf{qbqbqb} \cdots =$

$\mathsf{qq}^* = \cdots \mathsf{qqqqqq} \cdots =$



The patterns bq^* and qb^* differ only by translation and so we write $\mathsf{bq}^* = \mathsf{qb}^*$. Similarly, the pattern bb^* can be turned into qq^* by rotating the strip by 180°, so we have as well $\mathsf{bb}^* = \mathsf{qq}^*$ and thus there are only two different patterns.

If we repeat the above construction of the patterns, but allow the two-letter signature to be any ordered pair of aspects chosen from $\{\mathsf{b}, \mathsf{q}, \mathsf{d}, \mathsf{p}\}$, the number of possible signatures increases to 16. If we do not distinguish between patterns that can be obtained from each other by translations and rotations, we will find that there are six patterns.

$\mathsf{bb}^* = \mathsf{qq}^* = \cdots \mathsf{bbbbbb} \cdots =$

$\mathsf{bq}^* = \mathsf{qb}^* = \cdots \mathsf{bqbqbq} \cdots =$

$\mathsf{bd}^* = \mathsf{pq}^* = \mathsf{db}^* = \mathsf{qp}^* = \cdots \mathsf{bdbdbd} \cdots =$

$\mathsf{bp}^* = \mathsf{dq}^* = \mathsf{pb}^* = \mathsf{qd}^* = \cdots \mathsf{bpbpbp} \cdots =$

$\mathsf{dd}^* = \mathsf{pp}^* = \cdots \mathsf{dddddd} \cdots =$

$\mathsf{dp}^* = \mathsf{pd}^* = \cdots \mathsf{dpdpdp} \cdots =$

If, however, we do not distinguish between patterns which are mirror images of one another, then the first four complete the list.

The key observation is that we do not actually have to construct the strip patterns and observe them in order to determine how many different ones there are. Since the patterns are determined by the signatures, the method is to study what permutations of signatures do not change the pattern. The general model can be set up as follows.

We are given a set of permutations P that generates a group $\langle P\rangle$ which acts on the set of signatures $S = \{w_1, w_2, \ldots\}$, where each permutation in P transforms each signature into one that produces the "same" strip pattern, with the choice of group determining the definition of sameness. To count how many different strip patterns there are, we have to determine the number of orbits under the action of $\langle P\rangle$ on S. The perfect tool for counting the number of orbits is Burnside's lemma: The number of orbits equals the average number of points fixed by the permutations in the group.

More precisely, Burnside's lemma says that the number of orbits N of the group $\langle P\rangle$ acting on S is

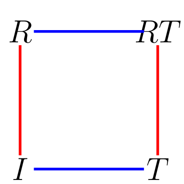
$$N = \frac{1}{|\langle P\rangle|} \sum_{p\in\langle P\rangle} |\mathrm{fix}(p)|$$

where $\mathrm{fix}(p)$ is the set of signatures fixed by the permutation p.

Burnside's lemma, also called the Cauchy-Frobenius lemma in the literature, has a long history, which can be found in [5, 11], but still has its place in advanced texts, e.g. see [10].

Let us say the group $\langle P\rangle$ is generated by two elements T and R. T interchanges the first and second elements of the signature, $T(XY) = YX$, and corresponds to a 1-unit horizontal translation of the strip pattern. R replaces each aspect with its rotated aspect and interchanges their order in the signature: $R(XY) = R(Y)R(X)$, where $R(\mathsf{b}) = \mathsf{q}$, $R(\mathsf{q}) = \mathsf{b}$, $R(\mathsf{p}) = \mathsf{d}$ and $R(\mathsf{d}) = \mathsf{p}$. R corresponds to a 180° rotation of the strip pattern.

The group $\langle P\rangle = \langle T, R \mid T^2 = R^2 = (TR)^2 = I\rangle$ is isomorphic to the Klein four group. Its Cayley graph is shown in Table 1 below: the group elements are represented as vertices. The horizontal edges correspond to multiplication by T, vertical edges to multiplication by R. In Table 1 we also show the action of the group $\langle P\rangle$ on the four signatures bb, bq, qb, qq, and see that there are a total of 8 signatures

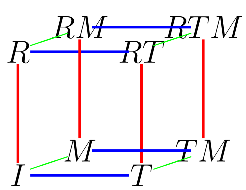


	I	R	T	RT
bb	bb	qq	bb	qq
bq	bq	bq	qb	qb
qb	qb	qb	bq	bq
qq	qq	bb	qq	bb

Table 1. $\langle P\rangle$ and its action on four signatures

fixed by elements of $\langle P\rangle$ (boxed). Since the group $\langle P\rangle$ has 4 elements, Burnside's lemma confirms the number of distinct patterns for $\langle P \rangle$ acting on signatures with two aspects to be $8/4 = 2$.

If we extend Table 1 to include 16 rows of signatures to account for all four aspects, we obtain a total of 24 signatures fixed by elements of $\langle P \rangle$, which are shown boxed in the first 4 columns, and so the formula in Burnside's lemma gives the number of distinct patterns as $24/4 = 6$. See Table 2.



	I	R	T	RT	M	RM	TM	RTM
bb	bb	qq	bb	qq	pp	dd	pp	dd
bq	bq	bq	qb	qb	pd	pd	dp	dp
qb	qb	qb	bq	bq	dp	dp	pd	pd
qq	qq	bb	qq	bb	dd	pp	dd	pp
bp	bp	dq	pb	qd	pb	qd	bp	dq
bd	bd	pq	db	qp	pq	bd	qp	db
qp	qp	db	pq	bd	db	qp	bd	pq
qd	qd	pb	dq	bp	dq	bp	qd	pb
pb	pb	qd	bp	dq	bp	dq	pb	qd
pq	pq	bd	qp	db	bd	pq	db	qp
db	db	qp	bd	pq	qp	dq	pq	bd
dq	dq	bp	qd	pb	qd	pb	dq	bp
pp	pp	dd	pp	dd	bb	qq	bb	qq
pd	pd	pd	dp	dp	bq	bq	qb	qb
dp	dp	dp	pd	pd	qb	qb	bq	bq
dd	dd	pp	dd	pp	qq	bb	qq	bb

TABLE 2. $\langle P' \rangle$ and its action on 16 signatures

To regard the strips as identical even after orientation reversing-transformations, we extend the group $\langle P \rangle$ by adding another generator, the mirror M, where M acts on signatures by $M(XY) = M(X)M(Y)$, where $M(\mathsf{b}) = \mathsf{p}$, $M(\mathsf{p}) = \mathsf{b}$, $M(\mathsf{q}) = \mathsf{d}$ and $M(\mathsf{d}) = \mathsf{q}$. This corresponds to taking the mirror image of the infinite strip in a horizontal line, and, together with the transformations we already have, allows us to consider strip patterns as identical if they differ by orientation-preserving as well as orientation-reversing transformations. Let $P' =$

$\{T, R, M\}$. The extended group $\langle P' \rangle$ has presentation

$$\langle T, R, M \mid T^2 = R^2 = M^2 = (TR)^2 = (TM)^2 = (RM)^2 = I \rangle$$

has 8 elements and is isomorphic to the direct product of three copies of the cyclic group on 2 elements. Table 2 shows also the Cayley graph of $\langle P' \rangle$ in which the three sets of mutually parallel edges correspond to multiplication by R, T and M, respectively. Table 2 shows the action of $\langle P' \rangle$ on the sixteen signatures; there are 32 signatures fixed by elements of $\langle P' \rangle$ (boxed).

Note: M mirrors the aspects in a horizontal mirror. We could have, alternately, used a vertical mirror M_V which mirrors aspects b and d, p and q; however the three groups generated by $\{T, R, M\}$, $\{T, R, M_V\}$, and $\{T, R, M, M_V\}$ are all the same since $M_V = MRM$, and $M = M_V R M_V$. Try to draw the corresponding Cayley graphs!

From Table 2 and Burnside's lemma, we obtain the result of $32/8 = 4$ different strip patterns with four motif aspects, confirming our earlier observation for the 'beadwork' pattern.

In fact, from the first four columns of Table 2, we can determine the previously computed number of patterns up to rotation and translation, with either all four aspects, all 16 rows, or just two aspects, the first 8 rows.

The main purpose of this note is to generalize the approach from the 1×2 case to the general case $1 \times n$, $n \geq 1$. The permutation groups become much more complicated and the sets of signatures on which they act grow much larger. To understand the general case it is enough to consider two relatively small representatives.

The 1×12 case

Let's compute the number of patterns arising from a strip of length 12 filled with choices from all four aspects, regarding patterns to be the same up to translation, rotation and reflection, that is, using the extended group, $\langle P' \rangle$.

To study the transformations of the signature, it is convenient to think of the signature as being drawn on the surface of a ring with 12 marked sections, such as in Figure 1 where the initial point in the signature is marked with a small triangle.

In fact, this is how you can create the strip patterns in practice; by inking the ring and then rolling out the pattern!

Any symmetric transformation of the ring clearly yields the the same pattern. Rotationally, the ring has dihedral symmetry, and the rotation group is generated by two rotations. The first is a rotation of 30° about

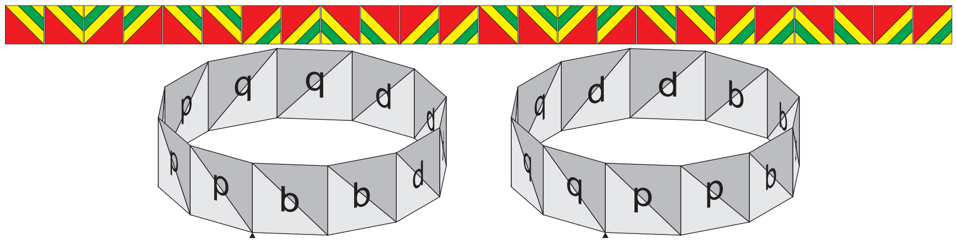


FIGURE 1. The signature $w =$ bbddbbppqqpp on a ring, and $T^4M(w)$, and their pattern. $T^6M(w) = w$, so $w \in \mathrm{fix}(T^6M)$.

the vertical axis through the center of the ring and corresponds to a translation of the strip pattern by one unit. We denote it by T:

$$T(a_1a_2\dots a_{12}) = a_2a_3\dots a_{12}a_1.$$

The second is a 180° rotation about the axis passing through the center

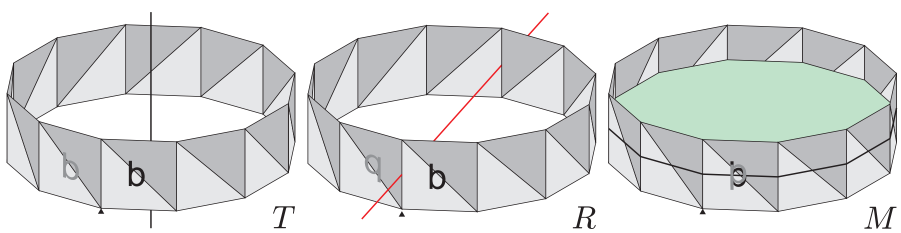


FIGURE 2. Transforming the signature on a 12-ring.

of the ring and passing through the midpoint of the initial boundary of the first motif, and corresponds to a 180° rotation of of the strip pattern. We denote it by R and its action on the signature is

$$R(a_1a_2\dots a_{11}a_{12}) = R(a_{12})R(a_{11})\dots R(a_2)R(a_1).$$

See Figure 2.

The elements T and R generate the dihedral group D_{12} in the usual way:

$$\langle P\rangle = \langle T, R \mid R^2 = T^{12}, RTR = T^{-1}\rangle$$

The orientation-reversing transformations can be added by adding the generator M which is a reflection in the horizontal plane that bisects the ring,

$$M(a_1\dots a_{12}) = M(a_1)\dots M(a_{12}),$$

and corresponds to a reflection of the strip pattern in a horizontal line; see Figure 2. We get the following presentation for $\langle P'\rangle$:

$$\langle T, R, M \mid T, R \mid I = R^2 = T^{12}, RTR = T^{-1}, TM = MT, RM = MR\rangle.$$

Of course, it is convenient to describe groups in terms of generators and relations, but that really doesn't help us in using Burnside's lemma, since we have to take the mean over *all* the elements of the group, not just the generators. Fortunately, at least for the dihedral group and its extension, we can easily visualize all the transformations. See Figure 3.

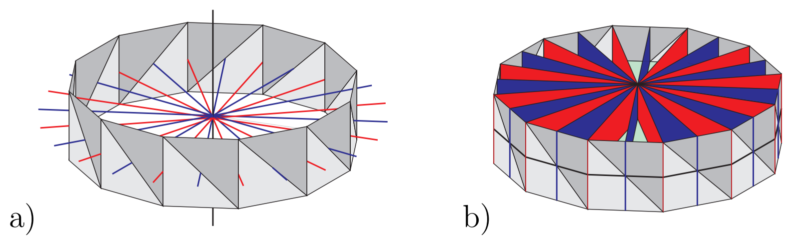


FIGURE 3. a) The axes of the rotational symmetries of the ring and b) the planes of the mirror symmetries, b.

All of the 24 transformations in $\langle P \rangle$ are rotational symmetries of the ring. There are 12 rotations of 180° around axes in the horizontal plane through the center of the ring, see Figure 3a. Of these, 6 have axes passing through the centers of two opposite motifs, and so fix no signatures since the motifs are asymmetric. The other six have axes on the midpoints of motif boundaries, with the motifs being divided into 6 pairs of orbits. So there are $6 \cdot 4^6$ fixed signatures for these transformations. See Figure 4, in which 6 independent choices

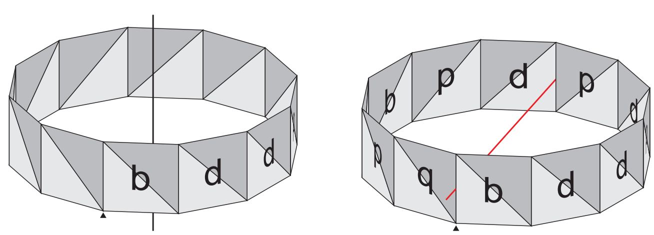


FIGURE 4. Creating a fixed signature for a horizontal axis rotation.

(b, d, d, p, b, q) for the first six positions in the signature yield the fixed signature bddpbqbqdppq.

The other 12 transformations in $\langle P \rangle$ are rotations about the vertical axis of $\frac{i}{12}360° = i \cdot 30°$, $i = 1 \ldots 12$. If i and 12 have a common divisor k, where $i = pk$ and $12 = qk$, then $q \cdot (i \cdot 30°)$ is a multiple of 360° and so this rotation has motif orbits of size a divisor of q. In fact, it is easy to see that the orbits of $i \cdot 30°$ are of size $12/\gcd(i, 12)$, and there

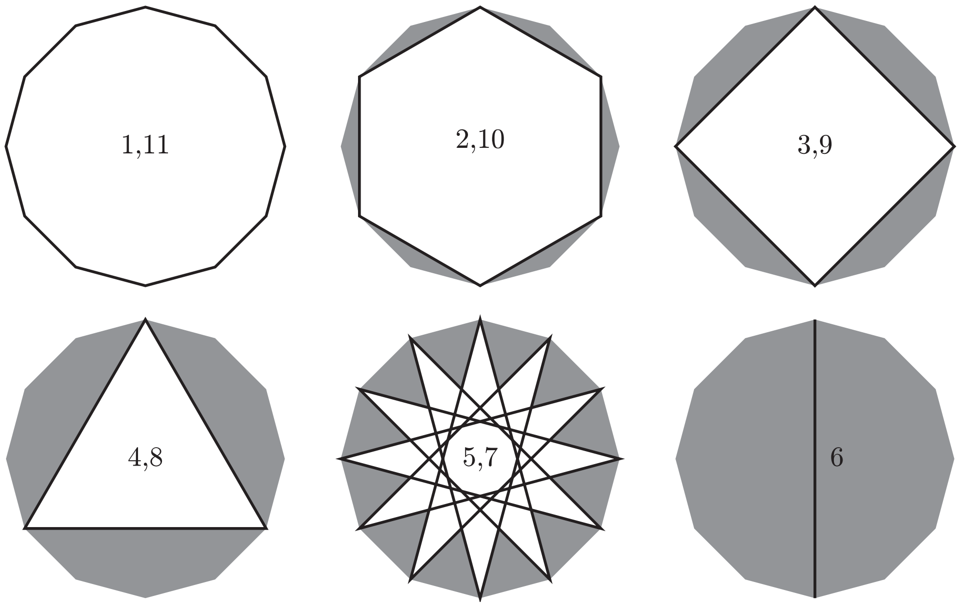


FIGURE 5. Orbits of rotations of a 12-gon.

are $\gcd(i, 12)$ of them. See Figure 5. So, for each divisor k of 12 there are rotations with aspect orbit sizes k. Each of these will have $4^{12/k}$ fixed signatures, since we are free to choose any of the four aspects for each orbit. See Figure 6 in which 4 independent choices (b, d, d, p)

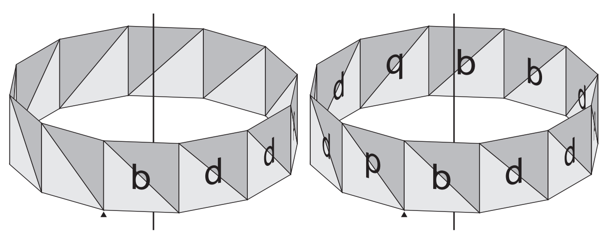


FIGURE 6. Creating a fixed signature for a vertical axis rotation.

for the first four positions in a signature and a $3 \cdot \frac{180^\circ}{12}$ rotation yield the fixed signature bddpbddpbddp. Twelve has divisors 12, 6, 4, 3, and 2. For 12 there will be 4 rotations with orbit size 12, corresponding to $i = 1, 5, 7, 11$, which is the number of positive integers less than 12 which are coprime to 12, giving $4 \cdot 4^1$ fixed signatures. For 6 there are two rotations of orbit size 6, $i = 2, 10 = 1 \cdot \frac{12}{6}, 5 \cdot \frac{12}{6}$, with 1 and 5 being the integers less than 6 coprime to 6; we get $2 \cdot 4^2$ fixed signatures. For 4 there are two rotations of orbit size 4, $i = 3, 9 = 1 \cdot \frac{12}{4}, 3 \cdot \frac{12}{4}$, with 1 and 3 being the integers less than 4 coprime to 4; we get $2 \cdot 4^3$ fixed signatures.

For 3 there are two rotations of orbit size 3, $i = 4, 8 = 1 \cdot \frac{12}{3}, 2 \cdot \frac{12}{3}$, with 1 and 2 being the integers less that 3 coprime to 3; we get $2 \cdot 4^4$ fixed signatures. For 2 there is one rotations of orbit size 2, $i = 6 = 1 \cdot \frac{12}{2}$, with 1 the only integer less than 2 coprime to 2; we get $1 \cdot 4^6$ fixed signatures. For 1 there is one rotation of orbit size 1, $i = 12$; we get $1 \cdot 4^{12}$ fixed signatures

The reader may recall that the number of positive integers less than k which are coprime to k is denoted by $\varphi(k)$ and is called the *Euler phi function.* So we have for the rotations about the vertical axis

$$\varphi(12)\cdot 4^{12/12}+\varphi(6)\cdot 4^{12/6}+\varphi(4)\cdot 4^{12/4}+\varphi(3)\cdot 4^{12/3}+\varphi(2)\cdot 4^{12/2}+\varphi(1)\cdot 4^{12/1}$$

fixed signatures.

For the 24 orientation-reversing symmetries, twelve are reflections in a vertical mirror, see Figure 3b. Six of these mirrors pass through the center of a motif, and so have no fixed signatures, while the other six mirros pass through the boundaries of the aspects, each of which has 6 aspect orbits of size 2 each, for a total of $6 \cdot 4^6$ fixed signatures. See Figure 7 in which 6 independent choices (b, d, d, p, b, q) for the first

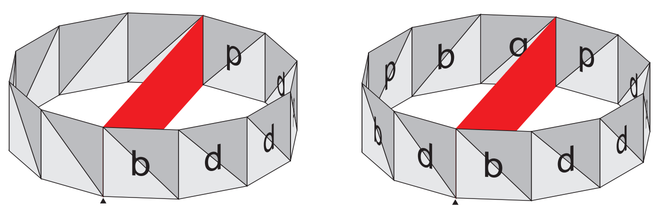


FIGURE 7. Creating a fixed signature for a reflection.

six positions in a signature yield the fixed signature bddpbqpdqbbd. The other 12 orientation-reversing transformations are not reflections at all, but are the product of one of the twelve rotations on the vertical axis with the reflection in the horizontal mirror, and are called *rotary reflections*. We have already analyzed the aspect orbits under these rotations. The only difference now is that, with the horizontal mirror, if the aspect orbit size is odd, specifically for $k = 3$ and 1, $(i = 4, 8, 12)$ there will be no fixed signatures since, following the aspect through its orbit, the aspect would return to its original position on the ring as a reflected aspect. For example, in Figure 8a we have chosen aspects b, b, and q respectively for the first three positions of a rotatory reflection of angle 90°, one for each of the three orbits, yielding the fixed signature bbqppdbbqppd. Trying the same method, Figure 8b, and choosing b, b, q, and b for the first four positions with the rotary reflection of 120°,

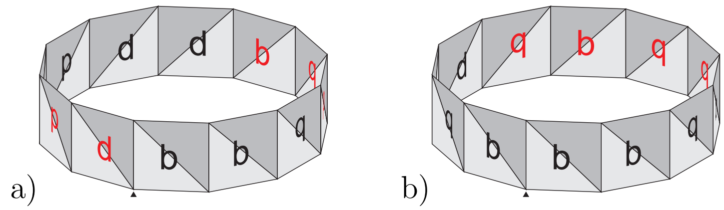


FIGURE 8. A fixed signature for a rotary reflection.

with rotational order 3, gives the signature bbqbppdpbbqb which is not a fixed signature under the 120° rotary reflection.

So, omitting the odd divisors, there are

$$\varphi(12) \cdot 4^{12/12} + \varphi(6) \cdot 4^{12/6} + \varphi(4) \cdot 4^{12/4} + \varphi(2) \cdot 4^{12/2}$$

signatures fixed by rotary reflections.

THE 1 × 15 CASE

For a ring of size 15, the rotations with vertical axis, and their rotary reflections are analyzed just as before, see Figure 9. Thus there are

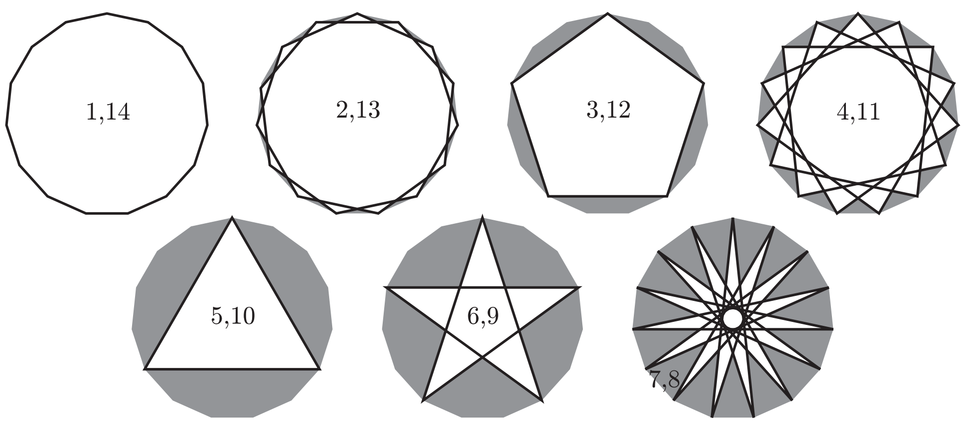


FIGURE 9. Orbits of rotations of a 15-gon.

$$\varphi(15) \cdot 4^{15/15} + \varphi(5) \cdot 4^{15/5} + \varphi(3) \cdot 4^{15/3} + \varphi(1) \cdot 4^{15/1}$$

fixed signatures for the first kind and no fixed signatures of the second because 15 has no even divisors.

The main difference here is that, since the ring is of odd size, every 180° rotation about a horizontal axis has one pole of the axis passing through the midpoint of an aspect boundary and the other passing though the center of the aspect, see Figure 10. None of these will have fixed signatures since the motif is assumed to be asymmetric. Similarly,

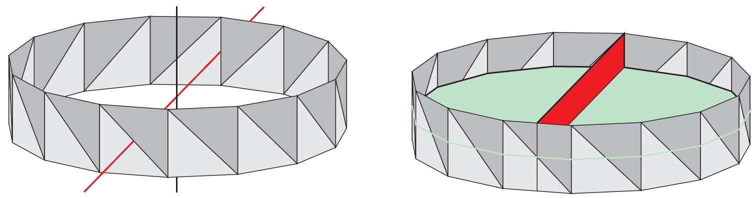


FIGURE 10. Axes and mirrors of a 15-ring.

each vertical mirror passes through an aspect boundary on one side of the ring, and passes through the middle of an aspect on the opposite side, again see Figure 10, so these also yield no fixed signatures because of the asymmetry of the individual motifs. Thus the total number of fixed signatures is $8 \cdot 4^{15/15} + 4 \cdot 4^{15/5} + 2 \cdot 4^{15/3} + 1 \cdot 4^{15/1}$.

THE $1 \times n$ CASE

In the general case, the group $\langle T, R, M\rangle$ has elements:

Vertical axis rotations	T^i
Horizonal axis rotations	$T^i R$
Vertical reflections	$T^i RM$
Rotary reflections	$T^i M$.

and acts on a signature $w = Q_1 Q_2 \cdots Q_n$, $Q_i \in \{\mathsf{b}, \mathsf{q}, \mathsf{d}, \mathsf{p}\}$ via

Translation:	$T(Q_1Q_2\cdots Q_n)$	$=$	$Q_2Q_3\cdots Q_nQ_1$.
Rotation:	$R(Q_1Q_2\cdots Q_n)$	$=$	$R(Q_n)\cdots R(Q_2)R(Q_1)$.
Mirror:	$M(Q_1Q_2\cdots Q_n)$	$=$	$M(Q_1)M(Q_2)\cdots M(Q_n)$.

For the vertical axis rotations the number of fixed signatures if there are two aspects is

$$v(n) = \sum_{k|n} \varphi(k) 2^{n/k},$$

while if there are 4 aspects the number is

$$V(n) = \sum_{k|n} \varphi(k) 4^{n/k}.$$

For horizontal axis rotations the number of fixed signatures if there are 2 aspects is

$$h(n) = \begin{cases} (n/2)2^{n/2} & \text{for } n \text{ even and} \\ 0 & \text{for } n \text{ odd} \end{cases},$$

while if there are 4 aspects the number is

$$H(n) = \begin{cases} (n/2)4^{n/2} & \text{for } n \text{ even and} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

For the orientation-reversing transformations, we are only considering the case with 4 aspects. For the vertical mirror reflections, there are $H(n)$ fixed signatures and, lastly, for the rotary reflections, there are

$$R(n) = \sum_{k|n,2|k} \varphi(k) 4^{n/k}$$

fixed signatures.

So, if we consider two aspects and rotational symmetry only, we have by Burnside's lemma

$$f(n) = \frac{v(n) + h(n)}{2n}$$

patterns.

If we allow four aspects but only consider rotational symmetry we have

$$F(n) = \frac{V(n) + H(n)}{2n}$$

patterns.

If we allow four aspects and consider mirror symmetry as well, there are

$$G(n) = \frac{V(n) + 2H(n) + R(n)}{4n}$$

patterns.

The number of orbits for each of the three cases, where $n = 1, \ldots, 30$ is given in Table 3. The sequence of numbers $f(n)$ appears as sequence with ID number A053656 in Sloane's On-Line Encyclopedia of integer sequences [8], where it is described as arising from the number of necklaces with n blue or red beads such that the beads switch color when the necklace is turned over, which is clearly equivalent to our situation. Our interpretation of $f(n)$ via the number of strip patterns is more naturally motivated than color switching-beads.

Note that $G(n) \approx 2F(n)$, which is reasonable, since the set of signatures is the same in both cases but the group $\langle P' \rangle$ that acts on it is double the size of $\langle P \rangle$. $G(n) = 2F(n)$ exactly when n is odd, in which case $R(n) = H(n) = 0$.

If n is large, then we expect that most signatures are asymmetric, and so will have orbit size $4n$. This would give us an approximate count of $G(n) \approx 4^n/(4n)$ which is necessarily an undercount since at least the signature $\mathsf{bbb}\ldots$ is symmetric. If $n = p$, a prime, then this is the only signature which is not in an orbit of size $4p$, so rounding up

n	$f(n)$ 2 motifs rotational group	$F(n)$ 4 motifs rotational group	$G(n)$ 4 motifs full symmetry group
1	1	2	1
2	2	6	4
3	2	12	6
4	4	39	23
5	4	104	52
6	9	366	194
7	10	1172	586
8	22	4179	2131
9	30	14572	7286
10	62	52740	26524
11	94	190652	95326
12	192	700274	350738
13	316	2581112	1290556
14	623	9591666	4798174
15	1096	35791472	17895736
16	2122	134236179	67127315
17	3856	505290272	252645136
18	7429	1908947406	954510114
19	13798	7233629132	3616814566
20	26500	27488079132	13744183772
21	49940	104715393912	52357696956
22	95885	399823554006	199912348954
23	182362	1529755308212	764877654106
24	350650	5864066561554	2932035552786
25	671092	22517998136936	11258999068468
26	1292762	86607703209516	43303860638644
27	2485534	333599972407532	166799986203766
28	4797886	1286742822580254	643371447241598
29	9256396	4969489243995032	2484744621997516
30	17904476	19215358696480536	9607679491405864

TABLE 3. Numbers of strip patterns under different notions of 'sameness'.

the rough approximation will give the actual value

$$G(p) = \frac{4^p + (p-1)4}{4p} = \frac{4^p}{4p} + \frac{p-1}{p}.$$

In any case, these are large numbers. In the case of $G(12)$, to scan over all the distinct patterns would take at least 14 days at a rate of one pattern per second and working 8 hours a day. For $G(15)$, the other case we examined, it would take more than a year.

Note also that the results are valid only in the case each motif is asymmetric: $R(X)$ and $M(X)$ are assumed to be distinct from X. We leave it to the reader to discover the formula for $f(n, m_a, m_s)$ for a more general case when there are m_a asymmetric motifs with $R(X)$ distinct from X and m_s symmetric motifs with $R(X) = X$.

The 1×4 case: Escher revisited

In Table 3, we see that $G(4) = 23$, which is exactly the number of different planar patterns that Escher found in answer to his original problem. The occurence of the same numbers is not a coincidence; in fact, the 1×4 strip pattern problem corresponds exactly to Escher's 2×2 problem.

In each case, there are four units that are filled with aspects of an asymmetric motif chosen from a set of four aspects. In our 1×4 case, the aspects are all obtained from aspect b by the action of a Klein-four group generated by the unit translation T and 180° rotation R. In Escher's case, the aspects were all obtained from aspect b by by the action of a cyclic group of order 4, generated by a 90° rotation.

Also, in each case, the group that acts on signatures for the patterns has order 16; it is a semi-direct product of a cyclic group of order 4 and a Klein four-group. In our 1×4 case, the cyclic group is generated by T, and the Klein four-group is generated by R and M. In Escher's case, the cyclic group was generated by a permutation induced by a 90° rotation of the 2×2 block, and the Klein four-group was generated by permutations induced by horizontal and vertical unit translations of the 2×2 block. If signatures for Escher's 2×2 blocks are written as a string $WXYZ$ that represents the aspects in a 2×2 block read in clockwise cyclic order, beginning with the upper left unit, then the action on signatures of the three generators for the "Escher group" is exactly the same as the action of T, R, and M on signatures in our 1×4 case.

Table 4 shows the $G(4) = 23$ strip patterns with their signatures ordered 'lexicographically', ($\mathsf{b} < \mathsf{q} < \mathsf{p} < \mathsf{d}$).

1	bbbb
2	bbbq
3	bbbp
4	bbbd
5	bbqq
6	bbqp
7	bbqd
8	bbpq
9	bbpp
10	bbpd
11	bbdq
12	bbdp
13	bbdd
14	bqbq
15	bqbp
16	bqbd
17	bqpd
18	bqdp
19	bpbp
20	bdbp
21	bdbd
22	bdqp
23	bdpq

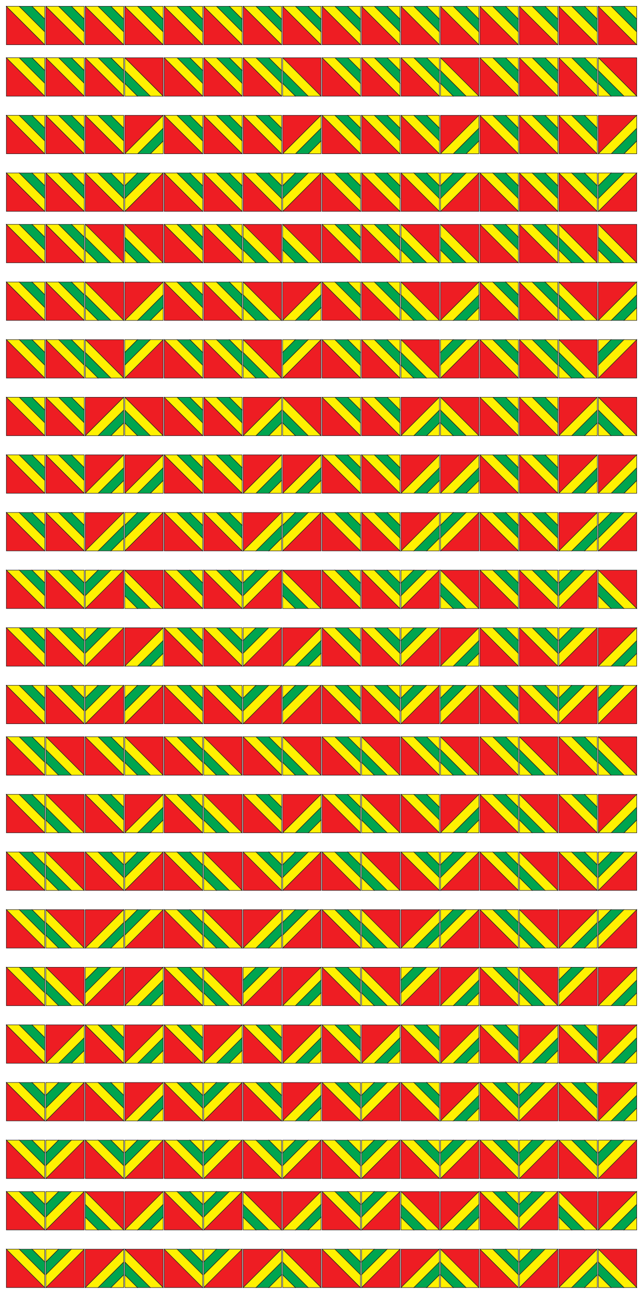


TABLE 4

The $f(4) = 4$ patterns with two aspects are in rows 1, 2, 5, and 14.

It is well-known that there are exactly seven symmetry groups of strip patterns. The notation for these groups are: 11 (translations only –

bb*); 12 (translations and 180° rotations – bq*); m1 (translations and vertical mirrors – bd*), 1g (translations and glide-reflections – bp*); mg (translations, 180° rotations, vertical mirrors, glide-reflections – bdpq*); 1m (translations and horizontal mirror – $\frac{\mathsf{bb}}{\mathsf{pp}}^{*}$); and mm (translations, 180° rotations, vertical mirrors, glide-reflections, and horizontal mirror – $\frac{\mathsf{bd}}{\mathsf{pq}}^{*}$).

All five without reflection symmetry in a midline mirror parallel to the edges of the strip occur in the patterns in Table 4. Most patterns have only translation symmetry. Here is the distribution of symmetry types:

Type 11	bb*	13 patterns
Type 12	bq*	3 patterns
Type 1g	bp*	2 patterns
Type m1	bd*	3 patterns
Type mg	bdpq*	2 patterns

Escher pursued several generalizations of his original problem, and these in turn have spawned many others: generalize to an $m \times m$ block with aspects chosen from a set of n aspects; generalize to higher dimensions; if the motif has under-over weave, allow inversion of over-under relationships to be a group operation; color the pattern so that overlapping strands do not share the same color; automate the pattern-creating process and pattern-coloring process. Many of these problems have been solved, and several are still under investigation. We list some published work on these problems in the references.

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