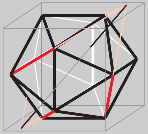

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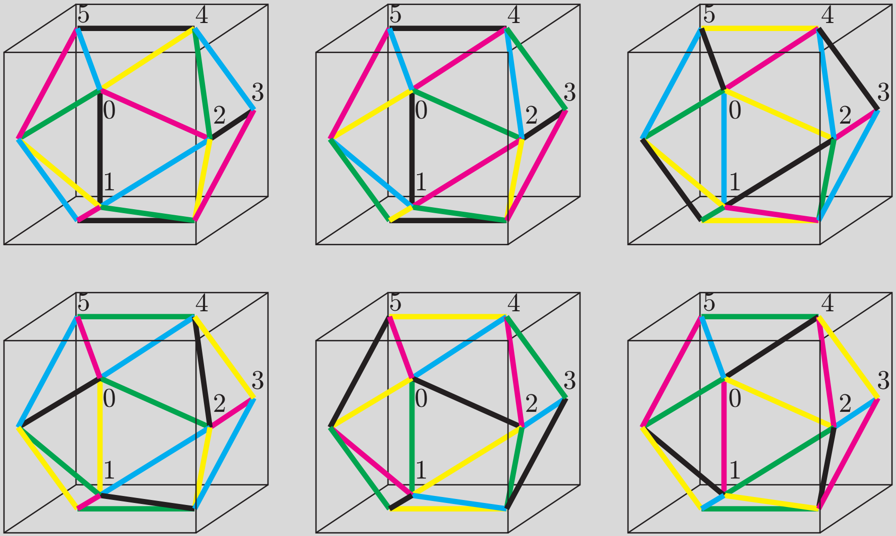
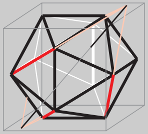
Constructing an Outer Automorphism for S_6

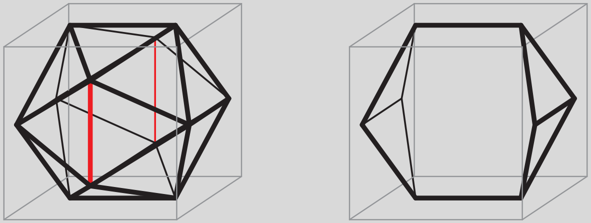
Except from: "The Cube Book"

December 2008

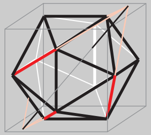


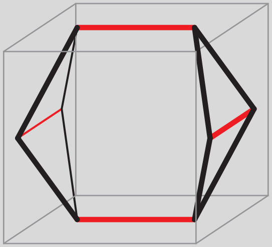


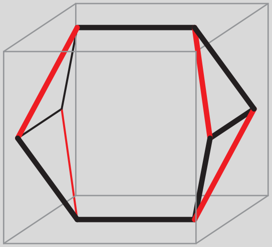
1. Perfect Matchings

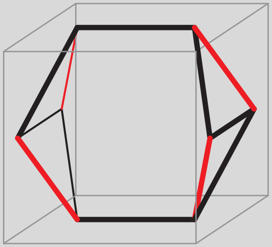


Constructing an antipodally symmetric perfect matching on the graph of the icosahedron.

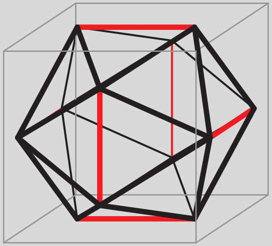


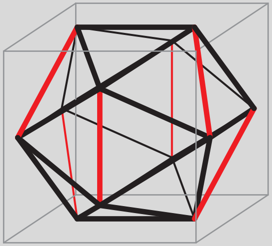


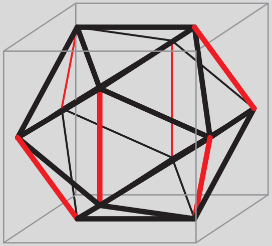




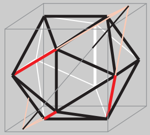
Completing a perfect matching on the graph of the icosahedron.

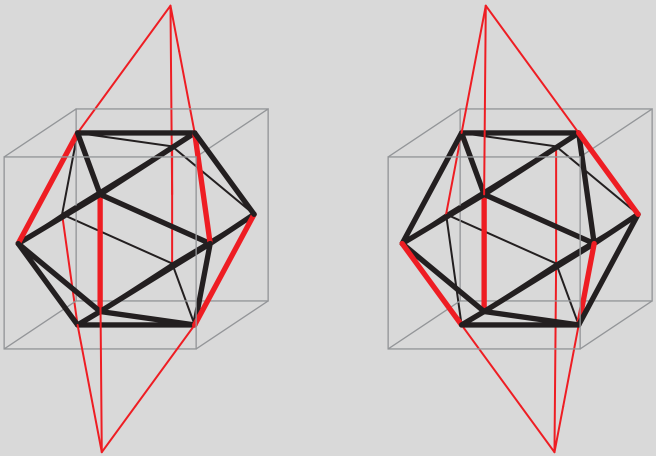




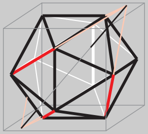


The three antipodally symmetric perfect matchings containing the back and front edges.
The first type is called *cubic*.

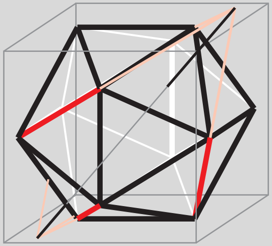




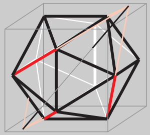
Joining the two points of coincidence gives a line which passes through the center of two antipodal triangles, and we call this type *axial*.



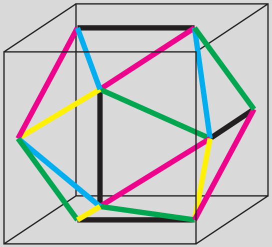
Another view of an axial matching:



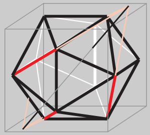
The edges of a cubic matching are disjoint from the edges of the axial matching with axis joining antipodal faces of the corresponding cube.



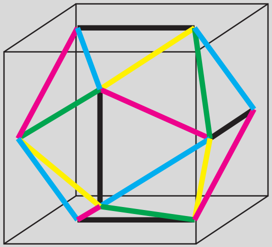
Combining axial and cubic matchings



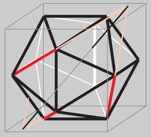
An edge partition into 5 antipodally symmetric perfect matchings.
1 cubic   4 axial



Another type of edge partition:



An edge partition into 5 antipodally symmetric perfect matching.
5 cubic



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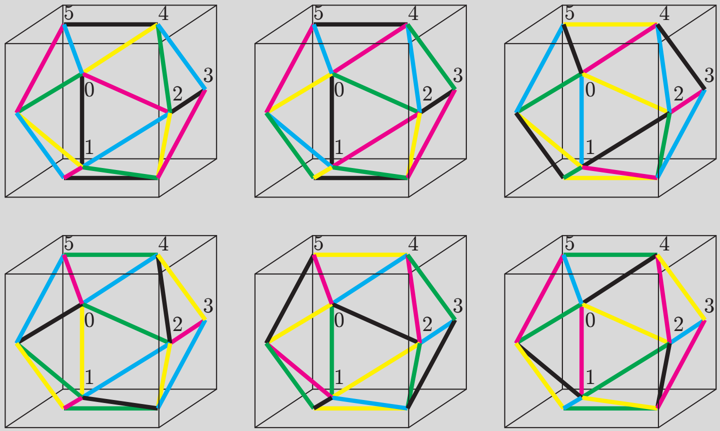
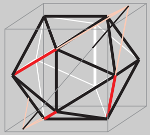
2. Labelling Partitions



black	01|23|45	01|23|45	05|12|34
blue	04|13|25	05|13|24	01|24|35
yellow	02|14|35	03|14|25	02|13|45
green	03|15|24	02|15|34	03|14|25
red	05|12|34	04|12|35	04|15|23

black	03|15|24	02|14|35	04|13|25
blue	04|12|35	04|15|23	05|14|23
yellow	01|25|34	03|12|45	02|15|34
green	02|13|45	01|25|34	03|12|45
red	05|14|23	05|13|24	01|24|35



3. Counting Partitions

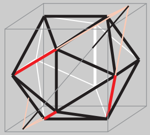
We count the number of possible partition labels:
5×3 arrays of pairs from $\{0, \ldots, 5\}$ such that

1. each digit $\{0, \ldots, 5\}$ occurs exactly once in each row;
2. each pair ij occurs exactly once in the array.

$$\begin{array}{ccc}
01|2i|jk & 01|2i|jk & 01|2i|jk \\
02|1j|ik & 02|1j|ik & 02|1j|ik \\
0i|??|?? & 0i|??|?? & 0i|1i|2k \\
0j|??|?? & 0j|??|?? & 0j|1k|2j \\
0k|??|?? & 0k|12|ij & 0k|12|ij
\end{array}$$

There are six ways to form the rows with 01 and 02.
Each can be uniquely completed to a legal label.

THEOREM: There are exactly 6 edge partitions into antipodally symmetric perfect matchings.



4. Permuting the labels

Every permutation

$$\{0, 1, 2, 3, 4, 5\} \longrightarrow \{0, 1, 2, 3, 4, 5\}$$

induces a permutation

$$\{\text{partitions}\} \longrightarrow \{\text{partitions}\}$$

So we have a homomorphism

$$S_6 \to S_{\{\text{partitions}\}}$$

PROPOSITION: The homomorphism $S_6 \to S_{\{\text{partitions}\}}$ is an isomorphism

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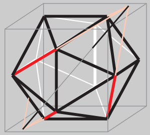
PROPOSITION: The homomorphism $S_6 \longrightarrow S_{\{\text{partitions}\}}$ is an isomorphism
PROOF: The kernel is S_6, A_6 or $\{1\}$.
So, if the kernel is non-trivial, every even permutation induces the identity on partitions.
Let's check (1 2 3):

black	01|23|45		01|23|45		05|12|34
blue	04|13|25		05|13|24		01|24|35
yellow	02|14|35		03|14|25	$\longrightarrow$	02|13|45
green	03|15|24		02|15|34		03|14|25
red	05|12|34		04|12|35		04|15|23
	$\downarrow$	$\nwarrow$		$\nwarrow$	$\downarrow$
black	03|15|24		02|14|35		04|13|25
blue	04|12|35		04|15|23		05|14|23
yellow	01|25|34	$\longrightarrow$	03|12|45		02|15|34
green	02|13|45		01|25|34		03|12|45
red	05|14|23		05|13|24		01|24|35

5. Automorphisms of S_6

We have an isomorphism: $S_6 \to S_{\{\text{partitions}\}}$
If we assign numbers to the partitions, we have an automorphism: $h : S_6 \to S_6$
And: $h(1\ 2\ 3) = (0\ 3\ 4)(1\ 2\ 5)$
Conclusion: h is not inner.

How rare is that?



THEOREM: If $n \neq 6$, every automorphism of S_n is inner.

PROOF:

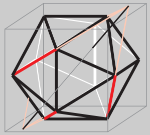
conjugation preserves the cycle structure of the permutations:

$$a_1 \xrightarrow{\sigma} a_2 \xrightarrow{\sigma} \dots \xrightarrow{\sigma} a_n \xrightarrow{\sigma} a_1,$$

a cycle of σ implies

$$\tau(a_1) \xrightarrow{\tau\sigma\tau^{-1}} \tau(a_2) \xrightarrow{\tau\sigma\tau^{-1}} \dots \xrightarrow{\tau\sigma\tau^{-1}} \tau(a_n) \xrightarrow{\tau\sigma\tau^{-1}} \tau(a_1),$$

is a cycle of $\tau\sigma\tau^{-1}$.



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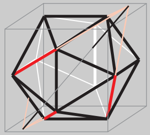
THEOREM: If $n \neq 6$, every automorphism of S_n is inner.

PROOF: (continued)

$f \in \mathrm{Aut}(S_n)$
f maps the conjugacy class of transpositions to a congugacy class of involutions.
f maps transpositions to the set of k disjoint transpositions, k fixed.
Transpositions: $\binom{n}{2}$
k disjoint transpositions: $\binom{n}{2k}(2k-1)(2k-3)\cdots(1)$
$\binom{n}{2} < \binom{n}{2k} \leq \binom{n}{2k}(2k-1)(2k-3)\cdots(1)$ if $2 < 2k < n-2$.



THEOREM: If $n \neq 6$, every automorphism of S_n is inner.

PROOF: (continued)

Cases: $2k - 2 = n$: $(2k-1)(2k-3)\cdots 1 = 1$, and so $k = 1$ and $n = 4$.
$2k - 1 = n$: $n = 2k + 1$

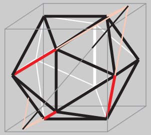
$$\binom{2k+1}{2} = \binom{2k+1}{2k}(2k-1)(2k-3)\cdots 1$$

$k = (2k-1)\cdots(1)$ so $k = 1$, with $n = 3$.
$2k = n$:

$$\binom{2k}{2} = \binom{2k}{2k}(2k-1)(2k-3)\cdots 1$$

so $k(2k-1) = (2k-1)(2k-3)\cdots 1$, or $k = (2k-3)\cdots 1$, and $k = 3$ and $n = 6$.

CONCLUSION: If $n \neq 6$, then f must map transpositions to transpositions.



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THEOREM: If $n \neq 6$, every automorphism of S_n is inner.

PROOF: (continued)

If $n \neq 6$, then f must map transpositions to transpositions.
Let h be an inner automorphism such that $h((0\ k)) = f((0\ k))$.
So $h^{-1}f(0\ k) = (0\ k)$ for all k.
$\{(0\ k)\}$ generates S_n.
$h^{-1}f$ is the identity
$h = f$.