Point Line Configurations and their Realizability

Brigitte Servatius
Zur Theorie der Netze und Configurationen von Konrad Zindler in Graz

- Elementary proof of a theorem of Möbius: 
  Given 4 points in the plane, one can, by ruler alone construct a point in the $\epsilon$-neighborhood of a given 5’th point for any $\epsilon > 0$.

- Generalization of Configuration: 
  A system of points and lines in the plane such that on every line there are at least 3 points and through every point there are at least 3 lines.
1. Zindler’s Construction
Realizable Moves

- Put a new point on a line.

- Put a new line through a point.

- Intersect two lines.

- Draw a line through two points.
- Join two components by putting a point of one component on a line of the other component.
Realizable Moves

- Put a new point on a line.
- Put a new line through a point.
- Intersect two lines.
- Draw a line through two points.
- Join two components by putting a point of one component on a line of the other component.
Dangerous Moves

- Intersect two lines.

- Draw a line through two points.
Realizable Moves on the Levi graph

- Add vertices of degree one.
- Add vertices of degree two such that bipartiteness and girth 6 are preserved.
  (between points of the same color a distance at least 4 apart.)
- Add edges between connected components (bridges).
Given a bipartite graph $G$ of girth 6, these moves may be reversed, provided there exists a vertex of degree at most 2. If $G$ is 3-regular, then $G - v$ can be built up from a vertex by allowable moves.
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Steinitz’s Theorem

Every symmetric $v_3$ configuration has a realization in the plane with at most one curved line.
Grünbaum’s Conjecture

Steinitz’s Theorem is true for configurations whose Levi graph is 3-connected.

Theorem

Steinitz’s Theorem is true for configurations whose Levi graph is 3-connected and edge 4-connected.
2. **Whiteley’s Theorem**

A generic picture in \( k - 1 \) space of an incidence structure lifts to a sharp scene in \( k \)-space if and only if

\[
i \leq a + kb - (k + 1)
\]

for all sub-incidence structures having at least two blocks.
For a 3-regular bipartite graph of girth six Whiteley’s count is violated by three.
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\[
\begin{align*}
  i = 6 & \quad p = 2 & \quad l = 6 & \quad l + 2p - 2 = 8 & \quad 2l + p - 2 = 12 \\
  i = 18 & \quad p = 8 & \quad l = 6 & \quad l + 2p - 2 = 20 & \quad 2l + p - 2 = 18 \\
  i = 24 & \quad p = 8 & \quad l = 9 & \quad l + 2p - 2 = 23 & \quad 2l + p - 2 = 24 \\
  i = 27 & \quad p = 9 & \quad l = 9 & \quad l + 2p - 2 = 25 & \quad 2l + p - 2 = 25 \\
  i = 3 & \quad p = 1 & \quad l = 6 & \quad l + 2p - 2 = 8 & \quad 2l + p - 2 = 12 \\
  i = 9 & \quad p = 8 & \quad l = 6 & \quad l + 2p - 2 = 20 & \quad 2l + p - 2 = 18 \\
  i = 21 & \quad p = 8 & \quad l = 9 & \quad l + 2p - 2 = 23 & \quad 2l + p - 2 = 24 \\
  i = 27 & \quad p = 9 & \quad l = 9 & \quad l + 2p - 2 = 25 & \quad 2l + p - 2 = 25 \\
  i = 30 & \quad p = 9 & \quad l = 9 & \quad l + 2p - 2 = 25 & \quad 2l + p - 2 = 25 
\end{align*}
\]
An \((8_4)\) spatial configuration.
\[ a = 8, \ b = 8, \ i = 32, \]
\[ a + 3b - 4 = 28 \]
A similar \((8_4)\) spatial configuration.

Levi graph is a hypercube

\[ a = 8, \ b = 8, \ i = 32, \]

\[ a + 3b - 4 = 28 \]