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The Geometry of Folding Paper Dolls

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The geometry of folding paper dolls

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When a parent sees a little girl sitting on the floor cutting paper dolls, many thoughts may come to mind: ‘She’s keeping out of trouble’ or ‘She’s making a mess’ or even ‘There go my tax returns’. The thought that should have come to my parent’s mind, however, was ‘One day she’ll be a mathematician’. My grandmother, who worked as a dressmaker, often allowed my sister and me to use her razor sharp scissors on the strips of leftover tracing paper. This paper is inspired by a notebook that I kept in grade school when I ‘studied’ paper dolls, and the figures are based on dolls found pressed between the pages.

The seven strip groups

Symmetry is an essential component of many traditionally female handicrafts: lacework, embroidery, weaving, hair braiding to name a few. The folding and cutting of paper dolls does not have the practical value of these crafts, but still may be used to illustrate some of the important tools of transformation geometry and crystallography. Japanese Origami, the epitome of paper art, seems to have discovered complex symmetry only recently [1].

Symmetry is here understood in the classical sense as being encoded by a transformation group of isometries. Strictly speaking, a string of paper dolls has very little symmetry, since dolls may be distinguished by their distance from the ends of the strip, however, it is more practical mathematically, and more consistent with artistic perception, to treat the string of dolls as if it is part of an infinite strip of dolls, see Figure 1.

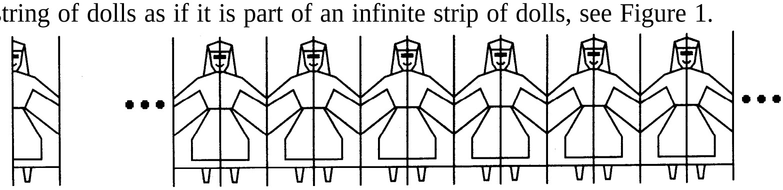

FIGURE 1 The classical paper doll pattern: pml1 = bdb

The isometries of the infinite strip are those isometries of the plane which preserve the strip, to whit:

- Translation parallel to the strip: b → b
- Rotation by 180° with pole on the centreline: b • q
- Reflection in the centreline: $\frac{b}{p}$ or in a line perpendicular to the centreline: b | d

- A glide reflection along the centreline: b → p

The choice of the letter b to show the action of the transformation is not accidental. The letter b has no symmetry in itself, and all four of its images under strip isometries occur as letters. The collection of all images of a set is called the *orbit* of the set. We can illustrate all the possible groups of symmetries of the strip by showing the orbit of a suitably placed b. We will also use these sequences to give us a handy mnemonic notation for these groups, which we will use in preference to the standard (but abstruse) international symbols given in parentheses. The international symbols have their origins in crystallography, see [2].

bbb – (p111) – cyclic, generated by a translation.

. . . b b b b b b b b b b b . . .

bdb – (pm11) – infinite dihedral group, generated by two parallel reflections perpendicular to the centreline.

. . . b d b d b d b d b d b . . .

bqb – (p112) – infinite dihedral group, generated by two 180° rotations with poles on the centreline.

. . . b q b q b q b q b q b . . .

bqp – (pma2) – infinite dihedral group, generated by a reflection perpendicular to the centreline, and a rotation with pole on the centreline.

. . . b q p d b q p d b q p d . . .

bpb – (p1a1) – cyclic, generated by a glide reflection.

. . . b p b p b p b p b p b . . .

$\frac{\text{bbb}}{\text{ppp}}$ – (p1m1) – generated by a translation and the centreline reflection.

. . . b b b b b b b b b b b . . .
p p p p p p p p p p p . . .

$\frac{\text{bdb}}{\text{pqp}}$ – (pmm2) – generated by the centreline reflection and two reflections in lines perpendicular to the centreline.

. . b d b d b d b d b d b . . .
. . p q p q p q p q p q p . . .

It is easy to see that these seven groups comprise all the symmetries of a strip. bbb and bdb are the only sequences of *equivalent* b's in which the top edge of the strip is preserved (all b's right side up), so $\frac{\text{bbb}}{\text{ppp}}$ and $\frac{\text{bdb}}{\text{pqp}}$ are the only possibilities containing a reflection in the centerline. If there is no centreline reflection, then the original b can be placed on the centreline and the b's form a simple sequence. If bbb is the sub-sequence of right-side up bbb's, then bqb and bpb are the two ways to shuffle the sequence bbb amongst itself upside down, while bqpd and bdpq, both equivalent to bqp, are the two ways to shuffle bdb amongst itself upside down. To see that these symmetries are geometrically distinct, the reader may verify that any pair may be distinguished by testing whether each

- preserves $+\infty$,
- preserves the top edge,
- preserves orientation, and
- contains a centreline reflection.

For a careful proof, see [2].

Our use of the sequences of letters above illustrates an important concept in geometric group theory, that of the fundamental region. If a group G acts by isometries on a subset *S* of Euclidean space, a *fundamental region* of *S* is any closed, simply connected region in which no interior point is fixed by an element of the group, a closed topological disc of the same dimension as *S*, which contains at least one element from each orbit of points in *S*, and such that no two interior points belong to the same orbit. A nice explanation is in [3].

As an example, consider the group of transformations of the plane generated by perpendicular glide reflections along arrows *A* and *B* in Figure 2.

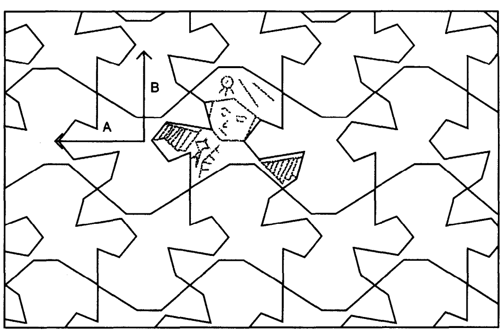

FIGURE 2 A fundamental region and its images

The portrait of Napoleon forms a fundamental region for this group of transformations, which is indicated by the fact that Napoleon's orbit tiles the plane. If a group of isometries has a fundamental region, then the group is said to act *discretely*, as contrasted with, say, the group of translations of a line by rational distances. For the discrete symmetries of the strip the fundamental regions can always be taken to be rectangles (boxes enclosing each "b"); however more creative fundamental regions may be devised in some cases. In general, the utility of the fundamental region is that its orbits define a tiling of *S*, and that tiling may be used to study the action of the group, see [4].

bdb

The group bdb corresponds to the classic method of cutting paper dolls. See Figure 3.

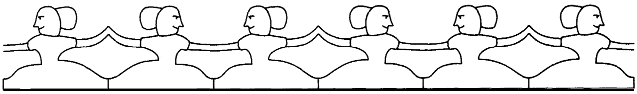

FIGURE 3 The classical paper doll pattern: bdb

The reason this works is that the parallel folds on the reflection lines arrange the paper so that points belonging to the same orbit lie above one another, hence any cutting preserves the orbits, hence also the symmetry. Our task is to devise similar methods for the other strip symmetries.

If the dolls face forward, then it is best to cut half a doll in the fundamental region, see Figure 1. As long as a strip of paper is left connecting the left and right edges, the pattern is connected.

bbb

The most direct way of producing dolls with only translation symmetry is to roll the strip of paper into a cylinder, hold it closed temporarily with clips or tape, cut a doll from the cylinder, and unroll. The paper making one circuit of the cylinder is the fundamental region of the strip. See Figure 4.

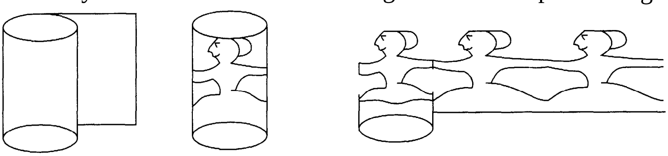

FIGURE 4 Translation by rolling: bbb

$\frac{\text{bdb}}{\text{pqp}}$ and $\frac{\text{bbb}}{\text{ppp}}$

These two cases can easily be obtained by first making a horizontal fold along the length of the strip, and proceeding as for bdb and bbb. See Figures 5 and 6.

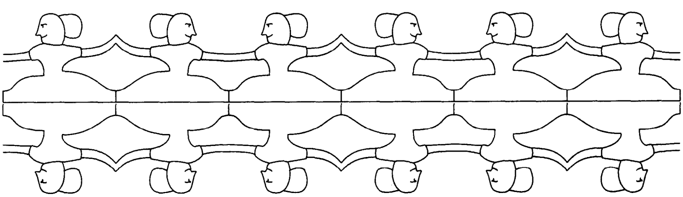

FIGURE 5 A pattern with centreline reflection: $\frac{\text{bdb}}{\text{pqp}}$

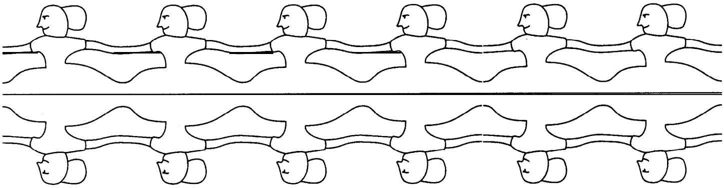

FIGURE 6 A pattern with centreline reflection: $\frac{\text{bbb}}{\text{ppp}}$

bqb

The group bqb is generated by 180° rotations, with poles equally spaced along the centerline, which presents a difficulty for paper folding since a 180° rotation cannot be directly achieved without disconnecting the paper. An indirect method is to use the fact that the product of two perpendicular plane reflections is a 180° rotation, and fold the strip as in Figure 7 into an

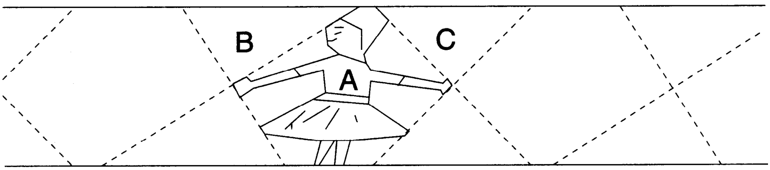

FIGURE 7 Folding gives 180° rotations

irregular hexagon, labelled *A,* with two right angles. Care must be taken, however, since this hexagon is ***not*** the fundamental region of the strip group. A true fundamental region consists of regions *A, B* and *C*. Regions *B* and *C*, however, are systematically folded inside the hexagon, so cutting a doll out of the hexagon and unfolding will yield a strip with symmetry bqb, but the dolls may have additions, see Figure 8, which can disturb the intended

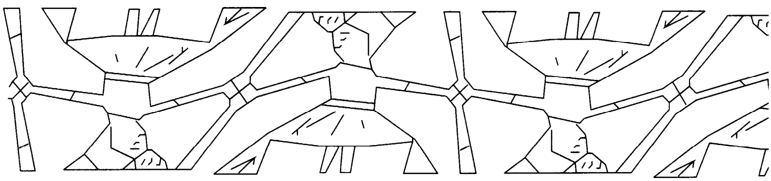

FIGURE 8 Dolls and partial dolls

design. (This is not a problem if you simply want abstract strip art.) Another problem is that it is possible to disconnect the strip inadvertently, even though the doll touches all six sides of the hexagon, as in Figure 9.

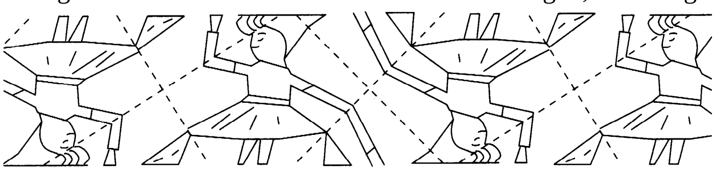

FIGURE 9 Disconnected dolls

You can ensure that the dolls are held together by having them touch the two rotation poles. Another disadvantage of this technique is that the dolls produced must be wider than they are tall. To avoid this problem, one can

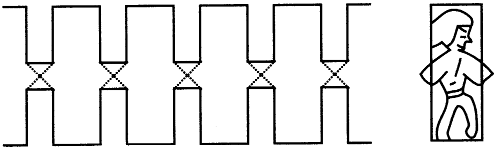

FIGURE 10 A practical solution to bqb

pre-cut the strip as in Figure 10, and fold only within the small attaching squares. Of course, the pre-cuts are an example of $\frac{\text{bdb}}{\text{pqp}}$ and so are easy to produce.

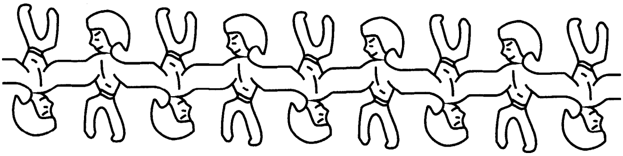

FIGURE 11 Results of method for bqb

bqp

The group bqp is generated by an alternating sequence of reflections and rotations, so we may use a combination of the techniques for bdb and bqb, see Figure 12.

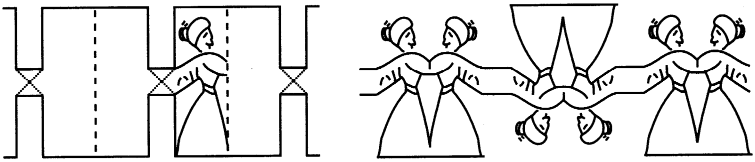

FIGURE 12 Method for bqp

bpb

The group bpb is generated by a single glide reflection along the centreline. The easiest method of achieving such a glide reflection is to roll the strip lengthwise, as in bbb, except that, to achieve the reflection, the strip must be rolled into a Mobius band. Twisting the band as tightly as possible, see Figure 13, we get, in the limit, the strip folded into an

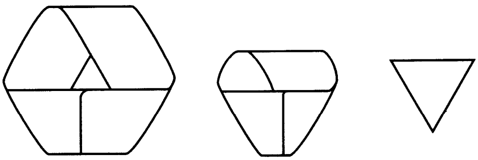

FIGURE 13 Cut on a Möbius Strip

equilateral triangle. Since this is the limiting case, any dolls made from cutting the Mobius strip and unrolling will have height to width ratio less than $\sqrt{3}/3$ – rather wide even for dolls with hoop skirts.

You can cut dolls directly on the limiting triangle, as in Figure 14. The

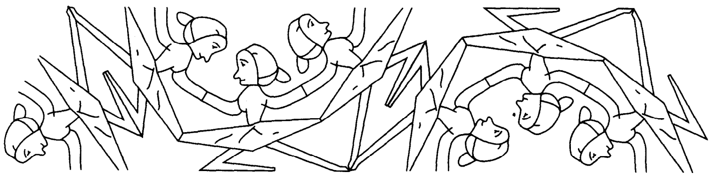

FIGURE 14 Three ballerinas to one fundamental region

unfolded strip will have symmetry bpb, but the symmetry acts on sets of three ballerinas, since three triangles make up the Mobius band, and hence the fundamental region.

A different technique that allows you to make taller dolls is to fold the strip to achieve a glide reflection, and then roll it into an ordinary cylinder. The product of a 180° rotation and a reflection not passing through the pole is a glide reflection in a direction perpendicular to the reflecting line:

b · q | p

We have seen how to approximate 180° rotations, so glide reflections can be approximated by folding the strip as in Figure 15, with or without

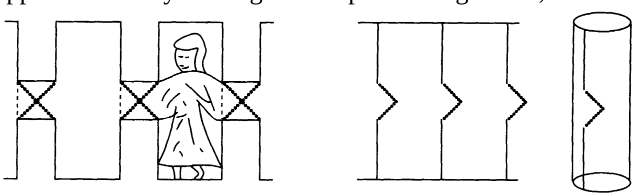

FIGURE 15 Folding to get bpb

precuts, and then rolling the strip into an ordinary cylinder such that the folded portions match up.

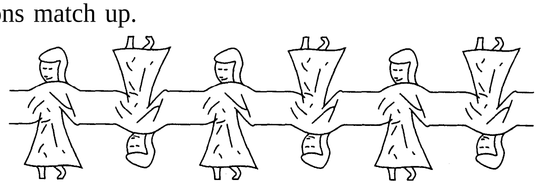

FIGURE 16 Better method for bpb

Subgroups

If you cut so that the fundamental region itself has non-trivial symmetry, then the symmetry exhibited by the unfolded strip may be larger than expected. This should be kept in mind when performing the more complicated constructions, since it is rather disappointing to end up with a strip that could have been more easily (and better) done as an ordinary bdb. How internal symmetries affect the strip group is really a question of which strip groups occur as subgroups of which others. This is a nice exercise I leave to you and your daughters.

Another aspect that touches on subgroups arises when the dolls are coloured. Instead of colouring all the dolls identically, one can also vary the colours symmetrically, that is, so that each symmetry induces a permutation of the colours. For instance, if the dolls of bbb have dresses coloured red and blue, alternating, then that colouring is symmetric. Symmetric colourings can be found by looking for normal subgroups of the symmetry group, see [5].

As a final word, I leave you with this puzzle. We know that groups generated by reflections are in general non-commutative, since two distinct reflections commute if and only if the reflecting lines are orthogonal. On the other hand, in the folding instructions for bpb we have not specified in which order the folds occur, and, indeed, the order of folding doesn't seem to matter. Or does it?

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