



Rigidity, connectivity and graph decompositions

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We say that a framework is *globally rigid* (in m -space) if all solutions to the system of quadratic equations obtained from requiring all edge lengths to be fixed, with the coordinates of the vertices as variables, correspond to congruent frameworks; we say that a framework is *rigid* (in m -space) if all solutions to the corresponding system in some neighborhood of the original solution (as a point in mn -space) come from congruent frameworks.

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0.1. Connectivity of graphs

A graph G is *connected* if it possesses a spanning tree, i.e. a subgraph $T = (V, E)$ such that $|E| = |V| - 1$ and

$$|F| \leq |V(F)| - 1 \quad \text{for all } F \subseteq E.. \quad (1)$$

Edge sets all of whose subsets satisfy inequality ?? are called *c-independent*. Here the prefix *c* indicates that this notion of independence is related to connectivity. A *cycle* is a minimally *c*-dependent edge set and if a cycle consists of n edges, it spans n vertices. We usually do not distinguish between the edge sets and the sub-graphs they induce, so sometimes we want to concentrate on the edges of a tree or a cycle, but we use the terminology for the entire subgraph when the meaning is clear.



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A graph is *2-connected* if for any two of its edges a and b there is a cycle C containing both a and b . Note that this is equivalent to the standard definition of (vertex) 2-connectivity, namely that G remains connected after removal of any one of its vertices. An edge which is not contained in any cycle of G is called a *c-bridge* of G .



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A connected graph is *edge-2-connected* if it remains connected after the removal of any edge. An edge-2-connected graph contains no bridges, i.e. every edge is contained in some cycle. The notion of edge-2-connectivity is weaker than 2-connectivity.



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The *c-rank* of an edge set E is the cardinality of a maximal independent subset of E . The *c-rank* of a connected graph $G(V, E)$ equals $|V| - 1$ and the *c-rank* of a disconnected graph equals the sum of the *c-ranks* of its connected components (maximal connected subgraphs). It is also equal to the number of bridges in G plus the sum over the ranks of the 2-connected components (maximal 2-connected subgraphs) of G .



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There are easy inductive constructions to generate all 2-connected graphs, namely G is 2-connected if and only if it can be built up from a cycle by sequentially adjoining edges (loops are not allowed) and subdividing edges. A graph is edge-2connected if and only if it can be built up from a vertex by adding edges (loops are allowed) and subdividing edges, see [?].



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0.2. Rigidity of graphs

A graph G is (generically) *rigid* (in the plane) if and only if it possesses a spanning *isostatic* subgraph, i.e. a subgraph $L = (V, E)$ such that $|E| = 2|V| - 3$ and we have

$$|F| \leq 2|V(F)| - 3 \quad \text{for all } F \subseteq E. \quad (2)$$

The inequalities (??) are called *Laman's condition*, and sets of edges E which satisfy it are called *r-independent*. Here the prefix r indicates that this notion of independence is related to rigidity.



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A *circuit* is a minimally r -dependent edge set and if a circuit consists of $2n - 2$ edges, it spans n vertices, see Figure ??



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A graph G is *2-rigid* if for any two of its edges a and b there is a circuit C containing both a and b . Note that this is *not* equivalent to requiring that G remains rigid after the removal of any one of its vertices, see [?]. An edge which is not contained in any circuit of G is called a *r-bridge* of G . A graph G is called *redundantly rigid*, if it is rigid and every one of its edges is contained in a circuit, or, equivalently, if it contains no r-bridge.



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The *r-rank* of an edge set E is the cardinality of a maximal independent subset of E . The *r-rank* of a rigid graph $G(V, E)$ equals $2|V| - 3$ and the *r-rank* of a non-rigid graph equals the sum of the *r-ranks* of its rigid components (maximal rigid subgraphs). It is also equal to the number of *r-bridges* in G plus the sum over the ranks of the 2-rigid components (maximal 2-rigid subgraphs) of G .



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There are several well known inductive procedures to generate rigid graphs. A graph is rigid if and only if it has a subgraph which can be obtained from an edge by a sequence of so called *Henneberg moves* [?, ?] see Figure ??, or edge addition (no loops allowed). A graph is 2-rigid if and only if it can be obtained from tetrahedra by a sequence of 1-extension, edge addition and 2-sum [?]. We describe the 2-sum in Section ??.



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1. Matroids on Graphs

A matroid $\mathfrak{M}(E, \mathcal{I})$ is a finite set E , the *ground set* together with a collection \mathcal{I} of subsets of E , called *independent sets*, such that the following three axioms are satisfied:

I1. $\emptyset \in \mathcal{I}$.

I2. If $E_1 \in \mathcal{I}$ and $E_2 \subseteq E_1$, then $E_2 \in \mathcal{I}$.

I3. If E and F are members of \mathcal{I} with $|E| = |F| + 1$, then there exists $e \in E \setminus F$ such that $F \cup e \in \mathcal{I}$.

A subset of E not belonging to \mathcal{I} is called *dependent*.



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The *rank function*, $\rho : 2^E \mapsto \mathbb{Z}$, is defined for $X \in E$ by

$$\rho(X) = \max(|F| : F \subseteq X, F \in \mathcal{I}).$$

The rank of the matroid \mathfrak{M} is the rank of the set E . A *base* of \mathfrak{M} is a maximal independent subset of E . A *circuit* of \mathfrak{M} is a minimal dependent subset of E , and a *bridge* of \mathfrak{M} is an element that belongs to every base of \mathfrak{M} .



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We are studying two matroids on the edge set E of a graph $G(V, E)$, namely the *cycle matroid*, $\mathfrak{C}(G)$, defined by its cycles as circuits (or, equivalently, by c-independent sets as the collection \mathcal{I}) and the (2-dimensional generic) *rigidity matroid*, $\mathfrak{R}(G)$, defined by r-independent edge sets.



1.1. The connectivity matroid $\mathfrak{C}(G)$

If $G(V, E)$ is a connected graph, the bases of $\mathfrak{C}(G)$ are the spanning trees of G . For any subset F of the edge set E , the rank of F in $\mathfrak{C}(G)$ is given by

$$\rho(F) = |V(F)| - c(F),$$

where $c(F)$ denotes the number of connected components of the subgraph of G induced by F . We stress that $\mathfrak{C}(G)$ is defined on the edge set E and the vertex set is not directly used in the definition of $\mathfrak{C}(G)$. In fact, there are non-isomorphic graphs with isomorphic connectivity matroids, for example all trees on the same number of vertices. In the case of trees the matroids are totally free, so this is not surprising. But even if G is 2-connected, the matroid information is not enough to uniquely determine the graph, however, if G is 3-connected, $\mathfrak{C}(G)$ determines G uniquely (up to isolated vertices), see for example [?].

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1.2. The rigidity matroid $\mathfrak{R}(G)$

It was first pointed out in [?] that r -independent edge sets as defined by ?? are the independent sets of a matroid. Moreover, a useful formulation of the rank function in terms of edge covers is given there: Let G_i be a cover of G by subgraphs G_i , with $n_i \geq 2$ vertices in each G_i , then the rank of $\mathfrak{R}(G)$ equals the minimum of $\sum_i (2n_i - 3)$ over all covers.



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If the edge set of G is r -independent, the cover may, at one extreme, be chosen to be the graph itself, or, at the other extreme, as $|E|$ singleton edges. Of course, other covers may work as well, see Figure ??.



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If the subgraphs G_i are not rigid the minimum may not be achieved even if G is rigid and independent and the edges of G_i partition E , see Figure ??.



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It is well known that in the plane (but not in higher dimensions), circuits are rigid, in fact they remain rigid after the removal of any single edge. The only possible cover to compute the rank of a circuit is the circuit itself.



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In order to analyze the structure of $\mathfrak{R}(G)$ we want to further examine decompositions of the edge set and their relation to the rank function.



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2. Matroid connectivity

Tutte [?] calls a matroid on the ground set E n -connected, if for any positive integer $k < n$ there is no partition of E into two sets E_1 and E_2 such that $|E_i| \geq k$ and $\rho(E_1) + \rho(E_2) \leq \rho(E) + k - 1$. With this definition every matroid is 1-connected.



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A matroid is 2-connected if there is no partition of E into two sets E_1 and E_2 such that $|E_i| \geq 1$ and $\rho(E_1) + \rho(E_2) \leq \rho(E)$, i.e. if it is not the direct sum of its restrictions to the E_i 's. It is clear that every matroid can be uniquely decomposed into a direct sum such that each of the summands is 2-connected. Note that many authors call a matroid *connected* if it is 2-connected in the Tutte sense. We choose to use Tutte's 2-connectivity, so that 2-connectivity of the graph G is equivalent to 2-connectivity of its cycle matroid $\mathfrak{C}(G)$.



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It is well known, see for example [?] or [?], that a matroid is 2-connected if and only if for any partition of the ground set into two sets, there is a circuit C intersecting both of them. In fact an even stronger conclusion holds, namely a matroid is 2-connected if and only if any pair of its edges is contained in a circuit.



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2.1. r -bridges and c -bridges

Let G be a connected graph. If $G = (V, e)$ is not edge-2-connected, then there is an edge $e \in E$, called a *bridge* whose deletion disconnects the graph, that is $r(E - e) = r(E) - 1$.



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Analogously, if $G = (V, E)$ is rigid but not edge-2-rigid, there is necessarily an edge $e \in E$ with $\rho(E - e) = \rho(E) - 1$ which we call an *r-bridge*. To avoid confusion we will refer to the standard bridges as c-bridges, see Figure ??.



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Being a bridge is a matroid property, so for both r -bridges and c -bridges we have:

Lemma 1 *The following are equivalent*

- 1. e is a bridge,*
- 2. e is contained in every basis,*
- 3. e is contained in no circuit.*



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Every edge of E is either a c -bridge, or is contained in a maximal edge-2-connected subgraph. Clearly we may use the c -bridges to identify the maximal edge-2-connected subgraphs

Theorem 1 *Let $G(V, E)$ be a connected graph with c -bridges B , and let $E - B$ be partitioned*

$$E - B = A_1 \amalg \dots \amalg A_k$$

into edge sets of the connected components of the subgraph of G induced by $E - B$. Then A_1, \dots, A_k induce the maximal edge-2-connected subgraphs of G .

Moreover, for each pair (i, j) , $1 \leq i < j \leq k$, there is a bridge $b_{i,j}$ such that A_i and A_j are contained in different connected components of $G - b_{i,j}$.



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PROOF: First we note that none of the subgraphs induced by the A_i 's contains a c-bridge, since that edge would be a direct summand of $\mathfrak{C}(A_i)$, and hence a direct summand of $\mathfrak{C}(E)$, and hence c-bridge for G . So each A_i is edge-2-connected, and is contained in the set of edges of some maximal edge-2-connected subgraph, D_i of G . Since D_i is bridgeless, we have $D_i \subseteq A_1 \amalg \dots \amalg A_k$, and, since it induces a connected graph, it is contained in one of the summands of the partition, so it must coincide with A_i .

To prove the second claim, choose a spanning tree T for G . T contains all bridges, and $T \cap A_i$ is connected for each i . There is a unique shortest path P in T connecting an endpoint of an edge in A_i to an endpoint of an edge in A_j . The path in T must contain bridges, since none of the A_i 's share a vertex. Any bridge contained in P will separate A_i and A_j . \square



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For $\mathfrak{R}(G)$ the statement is analogous.

Theorem 2 *Let G be a rigid graph with r -bridges B , and let $E - B$ be partitioned*

$$E - B = A_1 \amalg \dots \amalg A_k$$

into the edge sets of the rigid components of the subgraph of G induced by $E - B$. Then A_1, \dots, A_k induce the maximal edge-2-rigid subgraphs of G .

Moreover, for each pair (i, j) , $1 \leq i < j \leq k$, there is a bridge $b_{i,j}$ such that A_i and A_j are contained in different rigid components of $G - b_{i,j}$.



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PROOF: For the first statement, the argument is the same as in the proof of Theorem **??**, with prefix c replaced by prefix r .

For the second statement, let L be a spanning r -independent subgraph of G . The edges of L consist of the set of r -bridges as well as the edges of spanning r -independent subgraphs L_i for each of subgraphs induced by A_i . Consider P , an edge minimal rigid subgraph of L containing both L_i and L_j . P is actually the intersection of all the rigid subgraphs of L containing both L_i and L_j . Since $L_1 \amalg \dots \amalg L_k$ does not induce a rigid graph, P must contain some bridges, and the removal of any of these bridges from P , say b , has L_i and L_j in two distinct rigid components of $P - b$ by the minimality of P . Moreover, if $L - e$ had L_i and L_j in the same rigid component, then b would not be in the intersection of all rigid subgraphs contain these two sets. So A_i and A_j belong to two distinct rigid components of $L - b$. Thus A_i and A_j belong to two distinct rigid components of $G - b$. \square



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2.2. The 2-sum

The 2-sum, $M_1 \oplus_2 M_2$, of two matroids M_1 and M_2 , both containing at least 3 elements and having exactly one element e in common, where e is neither dependent (a loop) or a bridge in either of the M_i , is a matroid on the union of the ground sets of M_1 and M_2 excluding e and the circuits of $M_1 \oplus_2 M_2$ consist of circuits of M_i not containing e and of sets of the form $C_1 \cup C_2 \setminus e$ where C_i is a circuit of M_i containing e .

A matroid is 3-connected if and only if it cannot be written as a 2-sum.



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The 2-sum is also defined for graphs, but here one cannot identify two edges without specifying which pairs of endpoints are to be identified, in other words, without specifying an orientation on the edges to be amalgamated, see Figure ???. Note that the 2-sum of two cycles is a cycle.



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2.3. The 2-sum and 2-connectivity

Clearly the 2-sum of graphs is associative provided that the edges to be amalgamated are distinct, and so it is convenient to represent the result of a succession of 2-sums as a tree in which the nodes encode the graphs to be joined, and the edges encode the (oriented) edges to be amalgamated, see Figure ??.



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If all the graphs corresponding to the nodes in the amalgamation tree are 2-connected, then the graph which is the result of the joins encoded by the tree is also 2-connected. We consider the case when each of the graphs corresponding to the nodes in the tree is a *3-block*, that is, either 3-connected, a simple cycle with at least 3-edges, or graph consisting solely of two vertices and at least 3 parallel edges. If all the graphs corresponding to the nodes are three blocks with the restriction that no adjacent nodes correspond to cycles, and no adjacent nodes correspond to parallel edges, then the resulting 2-sum tree is called a *3-block tree*, see Figure ??.



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Tutte proved the following deep theorem characterizing finite 2-connected graphs, see [?, ?].

Theorem 3 *A 2-connected graph G is uniquely encoded by 3-block tree.*

This result has been generalized for matroids. Every 2-connected matroid has a unique encoding as a 3-block tree in which the 3-blocks are 3-connected matroids, bonds (matroids in which every 2-element subset is a circuit) and polygons (matroids consisting of a single circuit) [?] Theorem 18.



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It is worth noting that $\mathfrak{R}(G_1) \oplus_2 \mathfrak{R}(G_2) = \mathfrak{R}(G_1 \oplus_2 G_2)$ which follows easily from the fact that the 2-sum of cycles is a cycle.

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2.4. 2-sum and redundant rigidity

Redundantly rigid graphs in $\mathfrak{R}(G)$ are the analogs of edge-2-connected graphs in $\mathfrak{C}(G)$. For $\mathfrak{R}(G)$ we also have that the 2-sum of two circuits is a circuit, which was already observed in [?] and is also used in [?]. This means that if we consider a graph G with connected rigidity matroid $\mathfrak{R}(G)$, G is necessarily 2-connected and the 3-block decomposition of G and the matroid 2-sum decomposition of $\mathfrak{R}(G)$ as well as $\mathfrak{C}(G)$ yield identical 3-blocks. The requirement that $\mathfrak{R}(G)$ be connected restricts the kinds of 3-blocks occurring in the decomposition. Recall that a graph is globally rigid if it is both 3-connected and redundantly rigid.



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Theorem 4 *Let G be a rigid graph with connected rigidity matroid $\mathfrak{R}(G)$. Then the 3-blocks of G are multilinks or globally rigid graphs on at least four vertices.*

PROOF: If G is 3-connected it is globally rigid. If G is not 3-connected, we compute its 3-block tree T . Consider a leaf node G_L of T . G_L cannot be a multilink because G is simple, and it cannot be a cycle, because G is redundantly rigid. Therefore G_L is a 3-connected graph, which is redundantly rigid, hence globally rigid. Now the theorem follows by induction on the number of nodes of T . \square



misplaced?

Theorem 5 *A redundantly rigid 3-connected graph has a 2-connected rigidity matroid.*

PROOF: Suppose there are k 2-connected components of $\mathfrak{R}(G)$, $k > 1$. Since every edge is contained in a non-trivial circuit, each 2-connected component of the matroid induces a subgraph G_i which contains at least 4 vertices and, because of 3-connectivity of G , is attached to the rest of the graph by at least 3 vertices. So $|V| \leq (\sum n_i) - 3k/2$. The rank of a basis is the sum of the ranks of the components so a basis for the matroid will have $\sum(2n_i - 3)$ edges. The average valence for this basis will be

$$\frac{2|E|}{|V|} \geq \frac{2 \sum(2n_i - 3)}{(\sum n_i) - 3k/2} = 4$$

contradicting the Laman condition for a rigid graph. \square

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Theorem 6 *Let G be a graph with connected rigidity matroid. Let L be an spanning r -independent subgraph of G and \mathcal{C} the collection of fundamental circuits with respect to L . Then for every subset \mathcal{B} of \mathcal{C} the intersection $\mathcal{B} \cap \mathcal{C} \setminus \mathcal{B}$ is nonempty unless G consists of a single circuit.*

PROOF: Assume there is a set of fundamental circuits, \mathcal{B} , with respect to some basis L , and for each circuit $B \in \mathcal{B}$ we have $B = L_B + e_B$ for some $L_B \subseteq L$, intersecting none of the other fundamental circuits $\mathcal{C} \setminus \mathcal{B}$. Then $\rho(\bigcup_B B) = |\bigcup_B L_B|$ and similarly for the union of the fundamental cycles in the complement of \mathcal{B} , contradicting matroid connectivity. \square



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3. Graph decompositions via connectivity and rigidity

3.1. Connectivity Hierarchy

Given a graph $G(V, E)$, we can very naturally decompose it into connected components G_i (maximal connected sub-graphs) and it is clear that this decomposition is unique and induces a decomposition of the vertex set $V(G)$, as well as the edge set $E(G)$. Moreover $\mathfrak{C}(G) = \bigoplus \mathfrak{C}(G_i)$, which says that for a disconnected graph $\mathfrak{C}(G)$ is separable.



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Furthermore, we can ask for maximal edge-2-connected subgraphs. Each maximal edge-2-connected subgraph is clearly completely contained in a connected component and edge-2-connectivity induces a partition of the edge set of G into edge-2-connected components plus singleton edges, namely those edges of G that are not contained in a cycle. We call these the *trivial* edge-2-connected components. This partition is a refinement of the connectivity partition on the edges. V is not partitioned by edge-2-connected components. We can further refine the edge partition by asking for maximal 2-connected subgraphs, called *blocks*, since two blocks share again at most one vertex. The connectivity matroid is the direct sum over the blocks of G .



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3.2. Rigid Hierarchy

The maximal rigid sub-graphs of a graph G partition the edge set into direct summands of the rigidity matroid, which are called the rigid components of the graph. Similarly one can consider the maximal redundantly rigid subgraphs of G , see Figure ??.



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These redundantly rigid subgraphs together with the r -bridges also partition the edge set of G into direct summands of the rigidity matroid, necessarily a finer decomposition than the rigid components, and are called the *redundantly rigid components*. We consider r -bridges to be trivial redundantly rigid components, Figure ??.



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A redundantly rigid component, however, can be decomposed further if the corresponding restriction of the rigidity matroid is not 2-connected. The direct sum decomposition of the rigidity matroid into its 2-connected components is the finest decomposition we can obtain from the rigidity matroid information.



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3.3. Decomposition for redundantly rigid, but not 3-connected graphs

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Theorem 7 *If G is a redundantly rigid graph, its 3-blocks are either globally rigid, multi-links, or 3-connected rigid graphs containing r -bridges. The sum of the number of r -bridges over the 3-blocks of G equals the number of indecomposable summands in the direct sum decomposition of $\mathfrak{R}(G)$. PROOF: Let $G(V, E)$ be a redundantly rigid graph and suppose $\{u, v\}$ is a subset of V such that $G \setminus \{u, v\}$ is disconnected. Assume first that $(u, v) \notin E$. Then $G = G_1 \oplus_2 G_2$ where the 2-sum is taken along $e = (u, v)$ and both G_i are simple, redundantly rigid graphs and we can proceed by induction on the number of nodes of the 3-block tree of G . If $(u, v) \in E$, we know that $G \setminus e$ is rigid. Therefore at least one of the 2-summands, say G_1 must remain rigid after the deletion of edge(s) (u, v) , and $G_1 + e$ is redundantly rigid. If e is an r -bridge of G_2 , then $\mathfrak{R}(G) = \mathfrak{R}(G_1) \oplus \mathfrak{R}(G_2 \setminus e)$. So for each edge e in G that has as its endpoints a set of cut-vertices, we obtain two direct summands. \square*



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3.4. Decomposition for 3-connected graphs that are not redundantly rigid

If a 3-connected graph is not globally rigid, then it contains r-bridges. We now first decompose $\mathfrak{R}(G)$ into its 2-connected components. These components are either r-bridges or redundantly rigid graphs on at least four vertices whose rigidity matroid is connected. If they are 3-connected, they are globally rigid. If not, we compute their 3-block tree to write them as the 2-sum over globally rigid blocks and multilinks.

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The example in Figure ?? is 3-c-connected with the shaded pentagonal regions replaced with any of G_1, \dots, G_4 , and in fact 5-c-connected with the shaded pentagonal regions all replaced with G_1 's. Moreover, G is also rigid, so there is a single rigid component.

$\mathcal{R}(G)$ is not 2-connected, however, with each of the edges connecting pentagonal regions being singleton direct summands of $\mathcal{R}(G)$. In fact, if all pentagonal regions are replaced by G_4 , then G is isostatic and $\mathcal{R}(G)$ decomposes to the sum over singleton edges, so all the redundantly rigid components are trivial.

If the pentagonal regions are replaced by G_3 , then the two edges incident to the vertex of valence two in G_3 are also singleton summands of $\mathcal{R}(G)$, so that are a total of 27 r-bridges, as well as six summands corresponding to the six K_4 's which are redundantly rigid components.

If the pentagonal regions are replaced by G_2 , which is only 2-connected then further decomposition is necessary to determine the globally rigid pieces, which are the 3-r-blocks of G_2 , which coincide with the 3-c-blocks of G_2 .

If all pentagonal regions are replaced by G_1 , which is 3-connected, the redundantly rigid components are single edges, r-bridges, and the K_5 's which are 3-c-connected and 2-r-connected and hence all are globally rigid.



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4. Configuration Index

4.1. Definition and Examples

The *configuration index* $\iota(G, \mathbf{p})$ of a graph $G(V, E)$ whose vertices are embedded in the plane by $\mathbf{p} : V \rightarrow \mathbb{R}^2$ is the cardinality of the set of congruence classes of embeddings of G with the same edge lengths as in (G, \mathbf{p}) . We call \mathbf{p} *generic* if the coordinates of $\mathbf{p}(V)$ as point in $\mathbb{R}^{2|V|}$ are algebraically independent over \mathbb{Q} . If \mathbf{p} is generic, $\iota(G, \mathbf{p}) = 1$ exactly when G is globally rigid.

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The graph in Figure ?? is not globally rigid since it is neither 3-connected nor redundantly rigid, but its 3-block decomposition consists of two K_3 's and one K_4 (plus two 3-links), all globally rigid, so $\iota(G) = 4$ for all generic embeddings of G in the plane. If we remove the edge e shared by K_4 and K_3 in G the situation becomes more delicate. The 3-blocks are still globally rigid, now there is only one 3-link. However, the 2-sum along joins a redundantly rigid graph with an r-independent graph, the edge e is an r-bridge of one of the summands, S . Removing e from S yields a framework of degree of freedom 1. On the other hand, even though $K_4 - e$ is still rigid, removal of e destroys 3-connectivity as well as redundant rigidity and $\iota(K_4 - e) = 2$. For both possible realizations we have to see if there is a connected component of the configuration space of $S - e$. This might be the case for some, but not all the connected components, therefore the configuration index for G depends on the embedding, even if the embedding is generic.



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4.2. The configuration index of a graph with 2-connected rigidity matroid

Let G be rigid and let $\mathfrak{R}(G)$ be 2-connected. From Theorem ?? we know that its 3-blocks are globally rigid or multilinks, which makes it easy to compute their configuration index.

Theorem 8 *Let $G(V, E)$ be rigid, $|V| \geq 4$, and let $\mathfrak{R}(G)$ be 2-connected. If k is the number of globally rigid 3-blocks of $\mathfrak{R}(G)$ (which are not multi-links), then $\iota(G, \mathbf{p}) = 2^{k-1}$ for any generic embedding \mathbf{p} .*

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PROOF: Given an embedding of G , we can reflect its 3-blocks about axes determined by the endpoints of edges along which the 2-sum is taken, so 2^{k-1} is a lower bound for the configuration index of G . However, the 3-blocks are not necessarily subgraphs of G and the subgraphs of G induced by the vertex sets of the 3-blocks need not even be rigid or connected, they might in fact consist of isolated vertices. Let \mathbf{p} be a generic embedding of the vertices. All edge-lengths are in the algebraic closure of $\mathbb{Q}(\mathbf{p}(V))$ and the edge lengths of a base of $\mathfrak{R}(G)$ are also algebraically independent. Now if we prune a leaf F of the 3-block tree along e , then, since both 2-summands F and $G \setminus F$ are rigid after deletion of e , the length of e can be computed from the edge length information in either summand alone. $F - e$ might not be globally rigid, but since F is generically embedded, e will have different length in non-congruent embeddings. Any equality of the length of e in a re-embedding of F with the length of e in a re-embedding of $G \setminus F$ can be described as a non-trivial polynomial equation in the vertex coordinates, contradicting genericity. The theorem now follows by induction on k . \square



4.3. The configuration index of a redundantly rigid graph

If we consider a redundantly rigid graph whose rigidity matroid is not connected, the situation is much more complicated, since now the 3-blocks are not necessarily redundantly rigid. We can still use the 3-block decomposition of G to compute the configuration index.

Theorem 9 *Let $G(V, E)$ be a redundantly rigid graph with generic embedding \mathbf{p} and let B_i , $i = 1 \dots k$, be the 3-blocks of G which are not multi-links, ordered in such a way that B_i is globally rigid on at least for vertices for $1 \leq i \leq r$ and B_i is not redundantly rigid for $r + 1 \leq i \leq k$. Let D be the set of r -bridges contained in some B_i . Then*

$$\iota(G, \mathbf{p}) \leq 2^{k-1} \prod_{i=1}^k \iota(B_i) \prod_{e \in D} \left(\prod_{i=1}^r \iota(B_i \setminus e) \right).$$

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PROOF: Let e be an r -bridge of some B_i , such that $b \notin E$. G is the 2-sum along e and, since G is redundantly rigid and $e \notin E$, exactly one of the 2-summands, R , remains rigid after the removal of e and has some finite configuration index ι_e . Let d_1, \dots, d_i be the set of distances between the endpoints of e in all possible configurations of R . The other 2-summand has infinite configuration index after removal of e and its configuration space might have several connected components each of which determines a finite interval for the distance of the endpoints of e . If some, or all intervals contain a subset of the d_i 's, we get a realization of the 2-sum from the realizations of the parts. So the configuration index of G is bounded by the product of the configuration indices of the 2-summands. The theorem now follows by pruning the 3-block tree. \square



4.4. The configuration index of a 3-connected graph

Given a 3 connected graph $G(V, E)$, we know from Theorem ?? that its configuration index equals 1 if G is redundantly rigid. If G is not rigid, its configuration index is infinite, so we want to study 3-connected, rigid, but not redundantly rigid graphs. We first decompose $\mathfrak{R}(G)$ into 2-r-connected components, then decompose these further into 3-blocks, which are now all globally rigid. We first derive an easy lower bound on the configuration index in terms of r-bridges.

Theorem 10 *Let G be a rigid graph and B its set of r-bridges, then $\iota(G, p) \geq 2^{|B|}$.*

PROOF: Removal of an r-bridge b of G might or might not destroy 3-connectivity, but it certainly destroys rigidity. The configuration space of $G - b$ is, for any generic embedding p of G , a closed 1-dimensional manifold in $\mathbb{R}^{2|V|-3}$, so there are at least two distinct points on that manifold such that the distance between the endpoints is the length of e in $p(G)$. \square

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