Pow 2: Checkerboard Squares

Problem:

Find the total number of squares, $\boldsymbol{n_x}$, on an $\boldsymbol{x}$-by-$\boldsymbol{x}$ grid.

Process:

- ### method 1

To complete this problem, I did not start by finding the number of squares in an 8-by-8 grid. I thought that it would be easier to notice a pattern by starting with a 1-by-1 grid, then moving on to a 2-by-2 grid, and then continuing until I found ap pattern. At first, I thought that the pattern would resemble the pattern of the problem where an additional row of a triangle is added for each term of the sequence (1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, etc.), but this was not the case.

For a 1-by-1grid, it is clear that the number of squares, n, is 1. For a 2-by-2 square, I can break the problem into two parts because there are two types of squares that can be made: the 1-by-1 and 2-by-2 squares. There are 4 1-by-1 squares and 1 2-by-2 square. The summation of the number of 1-by-1 squares and the number of 2-by-2 squares is the total number of squares. Thus, $n_2 = 1 + 4 = 5$ for a 2-by-2 grid.

Moving to the 3-by-3 grid, I used the same strategy of breaking the problem into parts. This time I had to break it into three parts. One pattern is already evident. For the 1-by-1 grid, there was one part; for the 2-by-2 grid, there were two parts; for the 3-by-3 grid, there were three parts. Thus, for an $\boldsymbol{x}$-by-$\boldsymbol{x}$ grid, there will be $\boldsymbol{x}$ parts. There are 9 1-by-1 squares, 4 2-by-2 squares, and 1 3-by-3 square. Therefore, there $n_3 = 1 + 4 + 9 = 14$. Now another patern is evident. There are always x^2 1-by-1 squares, $(x-1)^2$ 2-by-2 squares, and $(x-2)^2$ 3-by-3 squares. If the pattern continued, there would be $(x-3)^2$ 4-by-4 squares, $(x-4)^2$ 5-by-5 squares, and $(x-(y-1))^2$ y-by-y squares where $x \geq y$.The number of squares (of a particular type of squares, not the total squares) is always a square number.

Next, to validate the conjecture above, I used the above equations to find n_4, and then I counted similarly to the method used to find n_3:

There will be 4 parts for a 4-by-4 grid (because for an $\boldsymbol{x}$-by-$\boldsymbol{x}$ grid, there will be $\boldsymbol{x}$ parts)

$$(4 - (1 - 1))^2 + (4 - (2 - 1))^2 + (4 - (3 - 1))^2 + (4 - (4 - 1))^2$$

$$30$$

Counting: There are 16 1-by-1 squares, 9 2-by-2 squares, 4 3-by-3 squares, and 1 5-by-5 square. Therefore, $n_4 = 16 + 9 + 4 + 1 = 30$.

The results came out the same; therefore, my conjecture is valid.

At this point, I could get the solution of this problem by doing $(8 - (1 - 1))^2 + (8 - (2 - 1))^2 \ldots \big((8 - (8 - 1))^2$. However, computing that would be very tedious. As the x becomes even larger, the task will get very tedious. To make the solution easier to retrive, I wrote the above conjecture in summation notation.

■ method 2

I also realized that I could count n_x differently. Starting with $x = 1$, $n_x = 1$. For $x = 2$, there will be the number of squares from the previous grid($n_x = n_{(x-1)}$) , plus the additional 1-by-1 squares, plus the additional 2-by-2 squares, ..., plus the additional y-by-y squares. Note: $x > 1$, $x \in \mathbb{Z}$. To find the pattern for this, i created a table (A_2 represents the additional 2-by-2 squares:

x	n_x	$n_{(x-1)}$	A_1	A_2	A_3	A_4	A_5
1	1	0	1				
2	5	1	3	1			
3	14	5	5	3	1		
4	30	14	7	5	3	1	
5	55	30	9	7	5	3	1

This pattern can be written as

$n_x = n_{(x-1)} + (2\,(x - 1) + 1) + (2\,(x - 2) + 1) + \ldots + (2\,(x - y + 1)$ (which is recursive) (note: doesn't work for $x = 1$ and $x = 2$).

Generalization:

▪ method 1

The summation notation function can be written as:

$$g[x\_] := \sum_{n=1}^{x} n^2$$

where *x* is the size of the grid (*x*-by-*x*).

Just to check:

```
g[2]
g[3]
g[4]
```

```
5

14

30
```

▪ method 2

The recursive summation function can be written as:

$$f[1] := 1$$

$$f[x\_] := f[x] = f[x-1] + \sum_{u=1}^{x} (2\,(x-u)+1)$$

where *x* is the size of the grid (*x*-by-*x*).

Just to check:

```
f[2]
f[3]
f[4]
```

```
5

14

30
```

proof

By finding the summation of A_1 to A_x (from method 2), the pattern for method 2 can be written as:

$$n_x = n_{(x-1)} + (2\,(n-1)+1) + (n-1)^2$$
$$n_x = n_{(x-1)} + (2\,n-2)+1) + \left(n^2 - 2\,n + 1\right)$$
$$n_x = n_{(x-1)} + n^2 - 2 + 1 + 1$$
$$n_x = n_{(x-1)} + n^2$$

Solution

method 1

```
g[8]
```

```
204
```

Thus, n_8=204

method 2

```
f[8]
```

```
204
```

Thus, n_8=204

■ more!

```
Table[{n, f[n]}, {n, 1, 15}] // TableForm
```

```
1   1
2   5
3   14
4   30
5   55
6   91
7   140
8   204
9   285
10  385
11  506
12  650
13  819
14  1015
15  1240

16  1496
17  1785
18  2109
19  2470
20  2870
21  3311
22  3795
23  4324
24  4900
25  5525
26  6201
27  6930
28  7714
29  8555
30  9455
```

Self-Assessment

Overall, I found the mathematics behind this POW easy. Method 1 was especially easy, and I thought of method 2 during the writeup. Every time that I typed a part of method 2, I would realize a new simplification (the proof), or thought of another way to express it. I didn't know that I was heading towards that: I just thought it would be nice to simplify. I think that I worked very diligently on this problem because I could have stopped at method 1, but I continued to method 2. One thing that I learned was that the sum of consecutive odd positive integers results in square numbers. Another thing that I learned was a few mathematica shortcuts: mainly for text and exponents.The only assistance I needed for this problem was gained by referring back to the mathematica labs (specifically recursions).