POF

Problem Statement

Given an $n \, x \, n \, x \, n$ tetrahedron that is cut such that there are $n - 1$ equally spaced cuts parallel to each of the three edges of each face, determine the number of tetrahedrons that have x exposed faces.

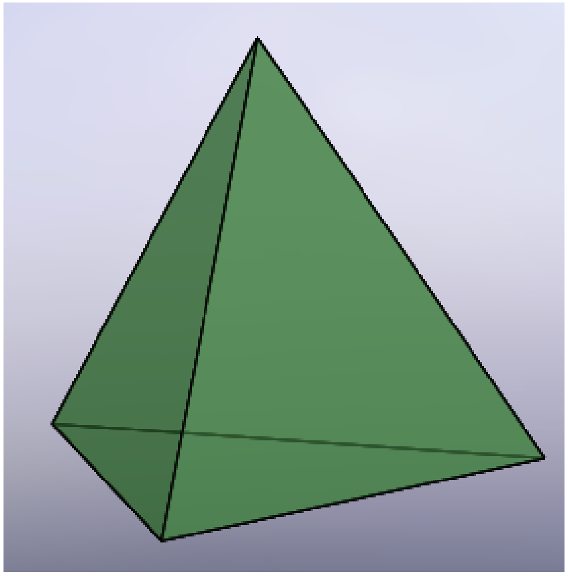
Process

I quickly realized that there are no tetrahedrons that have more than 4 painted faces (given any n). This is because a tetrahedron, by definition, has 4 faces.

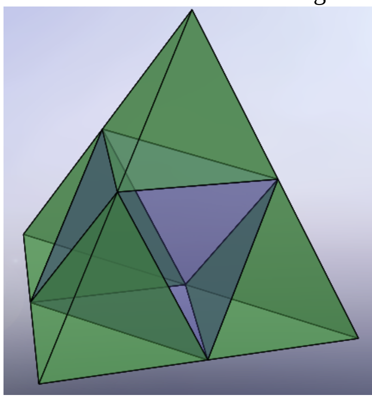
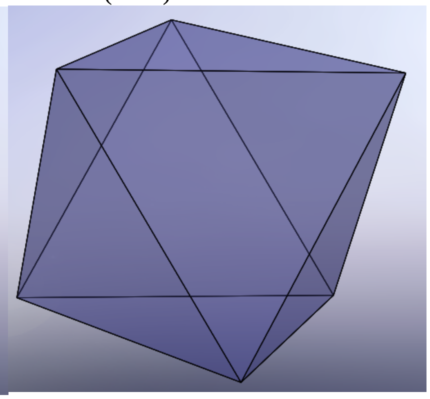
Definition: big tetrahedron = the $n \, x \, n \, x \, n$ tetrahedron; small tetrahedron = the small tetrahedron(s) produced by the cuts of the large tetrahedron.

To simplify the problem, I considered a regular tetrahedron, where all edges had a length of 1. The same solution would result from a non-regular big tetrahedron, and I could work faster with simpler numbers.

I first considered the $1 \, x \, 1 \, x \, 1$ big tetrahedron. There are $(1 - 1)$, or 0, cuts. It is clear to see that there is only one object, and that it has 4 exposed faces. Models were created with solidworks because *mathematica* doesn't have a nice tetrahedroid function (and i didn't want to write it); also, physical models are a pain. All models are shown in perspective without shadows.

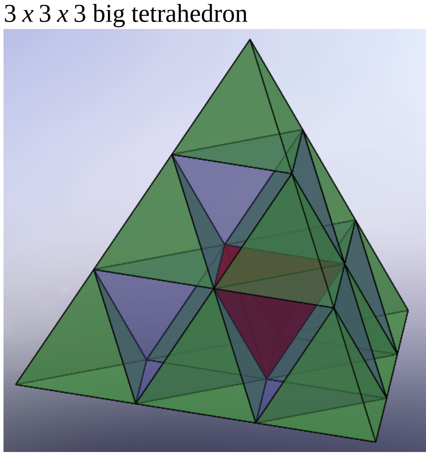
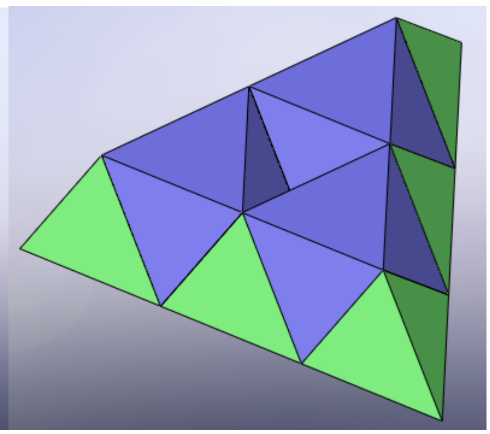


I next considered a 2 *x* 2 *x* 2 big tetrahedron. (1 cut).

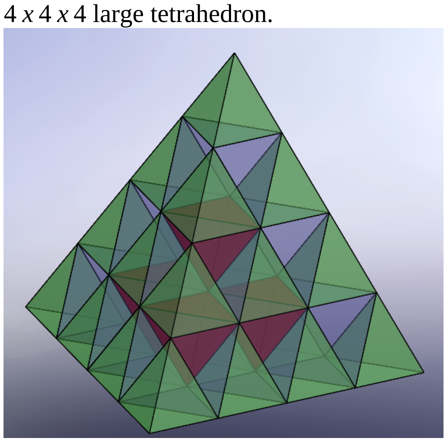
The first thing that I noticed was that there was empty space in the center of the tetrahedron after only construction the small tetrahedrons. The shape inside is a octahedron. There are 4 faces of this octahedron that are coincident with a face of a small tetrahedron, and there are 4 faces of this octahedron that are not flush against a face of another tetrahedron. Thus, there are 8 faces to this shape (so it is an octahedron). The octahedron is shown in purple and the small tetrahedrons are shown in green. There are 4 faces of the octahedron that are exposed. On each of the small tetrahedrons, there are 3 exposed faces because one face of each is touching a face of the octahedron. A picture of the octahedron alone is also shown.

3 *x* 3 *x* 3 big tetrahedron

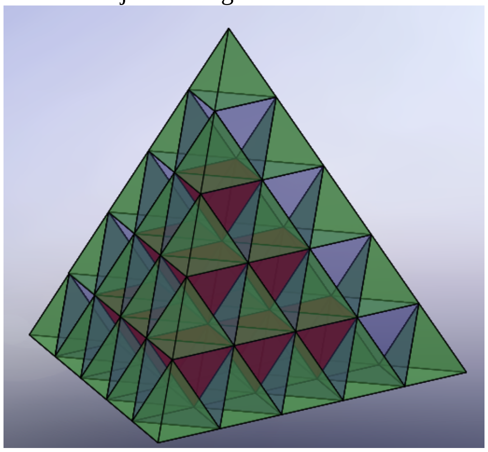



When adding another layer to the 2 *x* 2 *x* 2 large tetrahedron, I noticed there was another empty space. A congruent small tetrahedron, when rotated such that it is upside-down, fits into the empty space.

4 *x* 4 *x* 4 large tetrahedron.



Now I was just having fun....140 mates in Solidworks



Something can surely be said about the prevalence of triangular, tetrahedral, and pyramidal numbers in this problem. First one can see a sequence of triangular numbers be looking at the number of a type of object that is added to get the next *n*. For example, from a 3 x 3 x 3 to the next *n*, 4, one must add 10 upright small tetrahedrons (10 is a triangular number). The total number of objects added for the next row is always a tetrahedral number. The total number of objects is always a pyramidal number.

Solutions

For an *n x n x n* tetrahedron:

	4 exposed	3 exposed	2 exposed	1 exposed	0 exposed
1	1	0	0	0	0
2	1	4	0	0	0
3	0	8	6	0	1
4	0	8	18	4	4
5	0	8	30	16	11
6	0	8	42	36	25
7	0	8	54	64	49
8	0	8	66	100	86
9	0	8	78	144	139
10	0	8	90	196	211
11	0	8	102	256	305
12	0	8	114	324	424

Generalization

tetrahedrons with *exactly* 5 or more exposed faces: 0 always.

tetrahedrons with *exactly* 4 exposed faces: $\begin{cases} 1, & n = 1 \vee 2 \\ 0, & n > 0 \end{cases}$

tetrahedrons with *exactly* 3 exposed faces: When $n = 2$, there is a tetrahedron with 3 exposed faces at each of the corners of the large tetrahedron. When $n > 2$, the aforementioned tetrahedrons and the octahedron that are touching these tetrahedrons have 3 exposed faces. Thus: $\boxed{\begin{matrix} 0, & n < 2 \\ 4, & n = 2 \\ 8, & n > 2 \end{matrix}}$

tetrahedrons with *exactly* 2 exposed faces: On each edge of the larger tetrahedron, there are n small tetrahedrons that along this edge. There are always two corners along this edge because the edge is the line between the two vertices therefore, there are $n - 2$ small tetrahedrons along that edge. Each of these tetrahedrons has exactly 2 exposed faces. A tetrahedron has 6 edges; thus, the part of the total number of tetrahedrons that have exactly 2 exposed faces is equal to the product of the number of edges and the number of tetrahedrons on this edge with exactly 2 sides: $6(n - 2)$. In addition, the octahedron that touch exactly two of these tetrahedrons have 2 exposed faces: 6 (n - 3). Therefore, the total is: $6(n - 2) + 6(n - 3) = \boxed{6(2n - 5)}$

tetrahedrons with *exactly* 1 exposed faces: On each face of the large tetrahedron, there are n^2 faces (similar to square). It has already been shown that there are $(n - 2)$ small tetrahedrons per edge of the large cube. However, because many of the octahedron have been counted, I counted the middle (three rows in) area using side lengths of $(n - 3)$. Because there are 4 faces to a tetrahedron, there are always $\boxed{4(n - 3)^2}$ edge pieces.

tetrahedrons with *exactly* 0 exposed faces: There are three different types of objects, and at certain points in the sequence, a particular object will be fully enclosed. Because of the nature of the problem, the number of this particular type follows the tetrahedral number Sequence. The enclosed upside down tetrahedron sequence starts when $n = 3$, the enclosed upright tetrahedron sequence begins when $n = 5$, and the enclosed octahedron sequence begins when $n = 6$. Tetrahedral numbers are simply the sum of triangular numbers. Earlier this year, we used finite differences to find a closed form for this relation: $\frac{(n)(n+1)(n+2)}{6}$. By applying this to the current situation, where the numerator has been updated to reflect position in the sequence (e.g. $n = n - 3$ for when the sequence starts at $n = 3$), I get:

$$\boxed{\frac{((n-5)(n-4)(n-3)+(n-4)(n-3)(n-2)+(n-2)(n-1)(n))}{6}}$$

Simply another way to write this (yay combinatorics!): $\boxed{\sum_{n=3}^{n} \frac{(n-1)!}{(n-3)!\,2!} + \sum_{n=5}^{n} \frac{(n-3)!}{(n-5)!\,2!} + \sum_{n=6}^{n} \frac{(n-4)!}{(n-6)!\,2!}}$

Self-Assessment

Overall, I am happy with the outcome of this POF. That being said, I don't feel like I put my full potential into it because I did the entire problem in a day. I have been so busy during the past week with a robotics competition, a wedding, stream b meetings, and science fair. Honestly, spending time for my science fair was much more important than this problem. If I had more time, I would have really liked to do the graphics in mathematica instead of Solidworks.