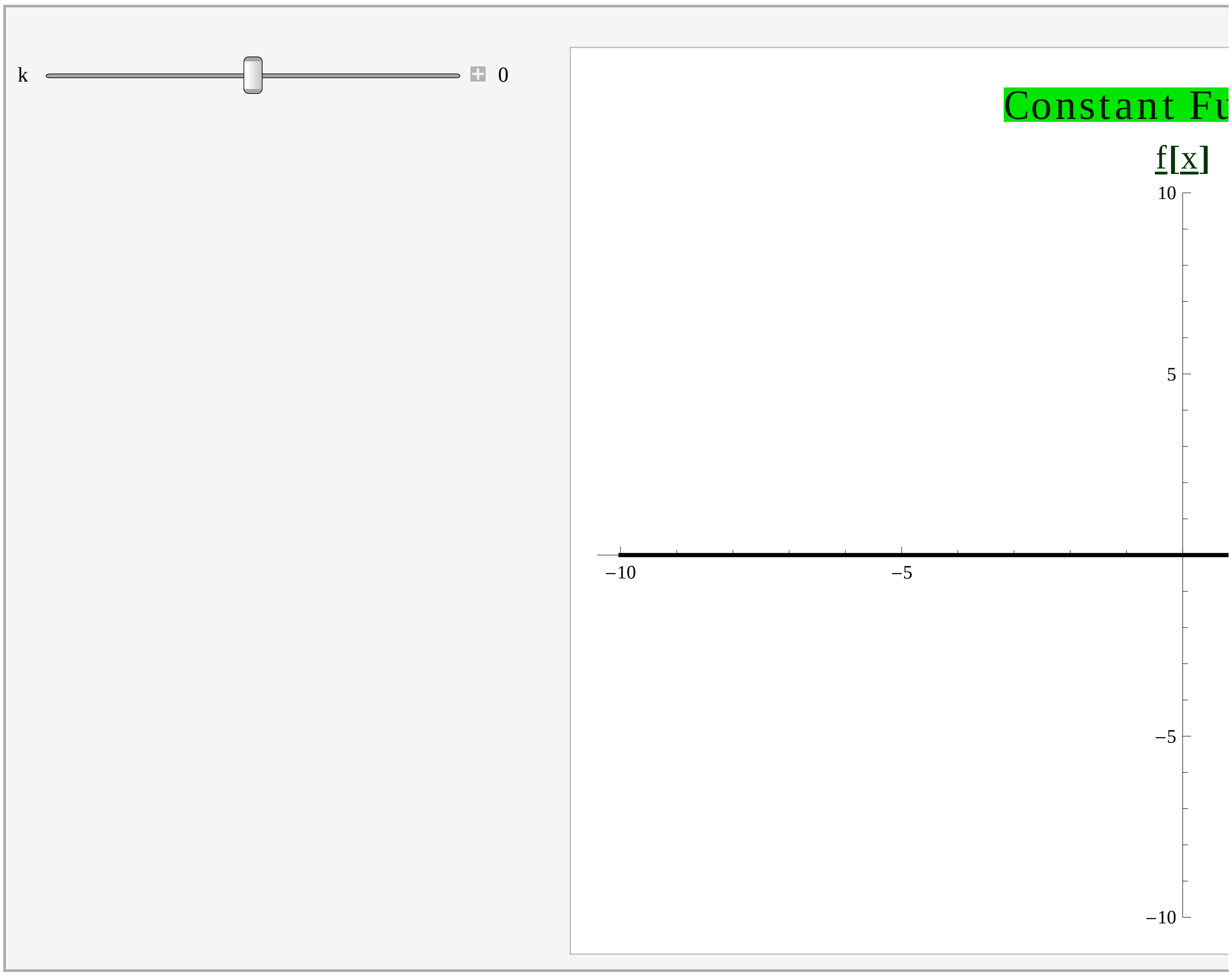
Transformations of Functions

Toolbox Functions

Constant function $f[x] = k$

Graph

Out[106]=



Domain

The domain of the function is ℝ because the graph is continuous.

Range

The range of this function is (k, k) because the graph does not have a slope.

Intervals of Increasing/Decreasing

The function is not increasing or decreasing on any interval because the function does not have any slope.

Intervals of Concavity

The function does not have any concavity on any interval because the function does not have any slope.

Parity

Testing even parity:

$$f[x] = f[-x]$$
$$k = k$$
$$\text{True}$$

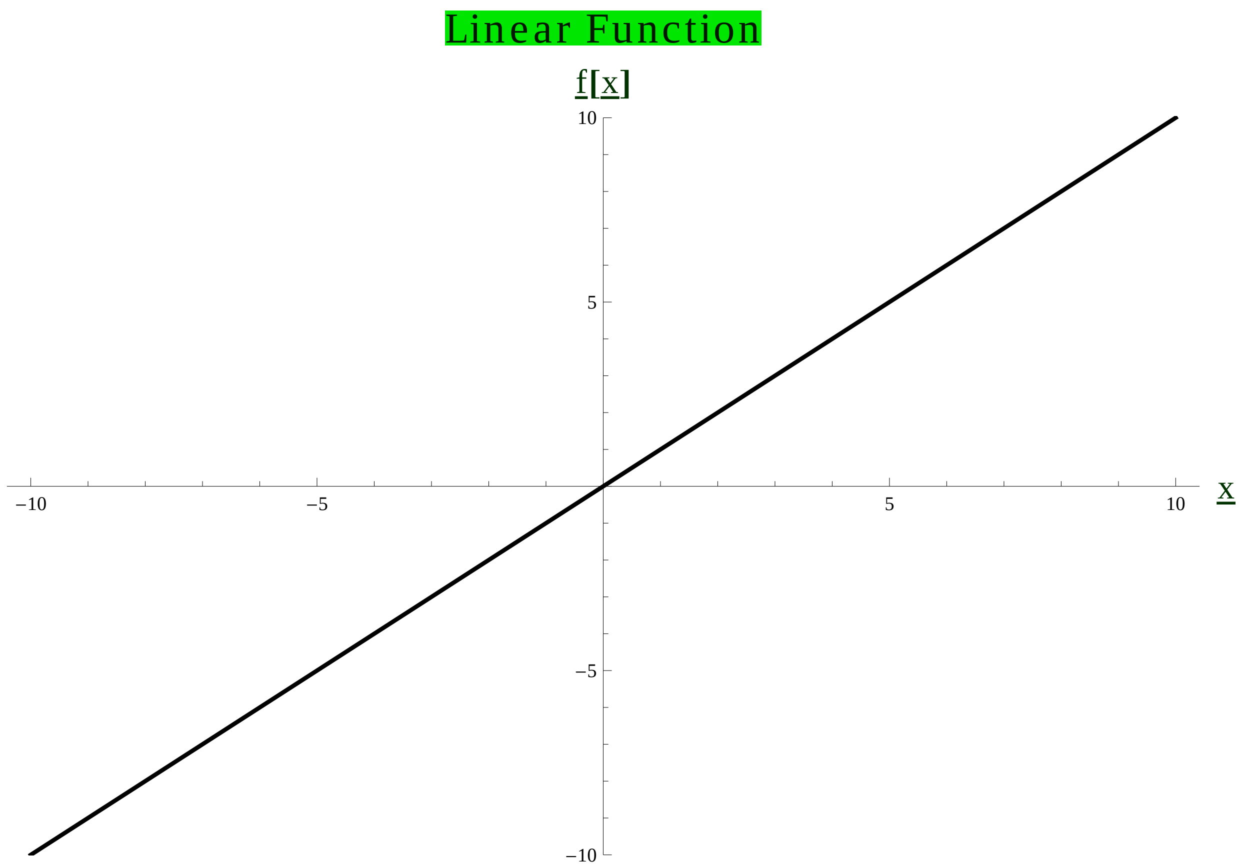
Testing odd parity:

$$-f[x] = f[-x]$$
$$-k = k$$
$$\text{False}$$

Interestingly, the function is odd when $k = 0$. However, the function cannot be considered odd because it is not odd for *all* values of k.

Linear function $T[x] = x$

Graph



Domain

The domain of the function is ℝ because the graph is continuous.

Range

The range of the function is ℝ because the graph is continuous and has a constant slope.

Intervals of Increasing/Decreasing

The function is increasing for ℝ because the function has a positive slope for the entire interval.

Intervals of Concavity

The function does not have any concavity on any interval because the function does not have any change in slope.

Parity

Testing even parity:

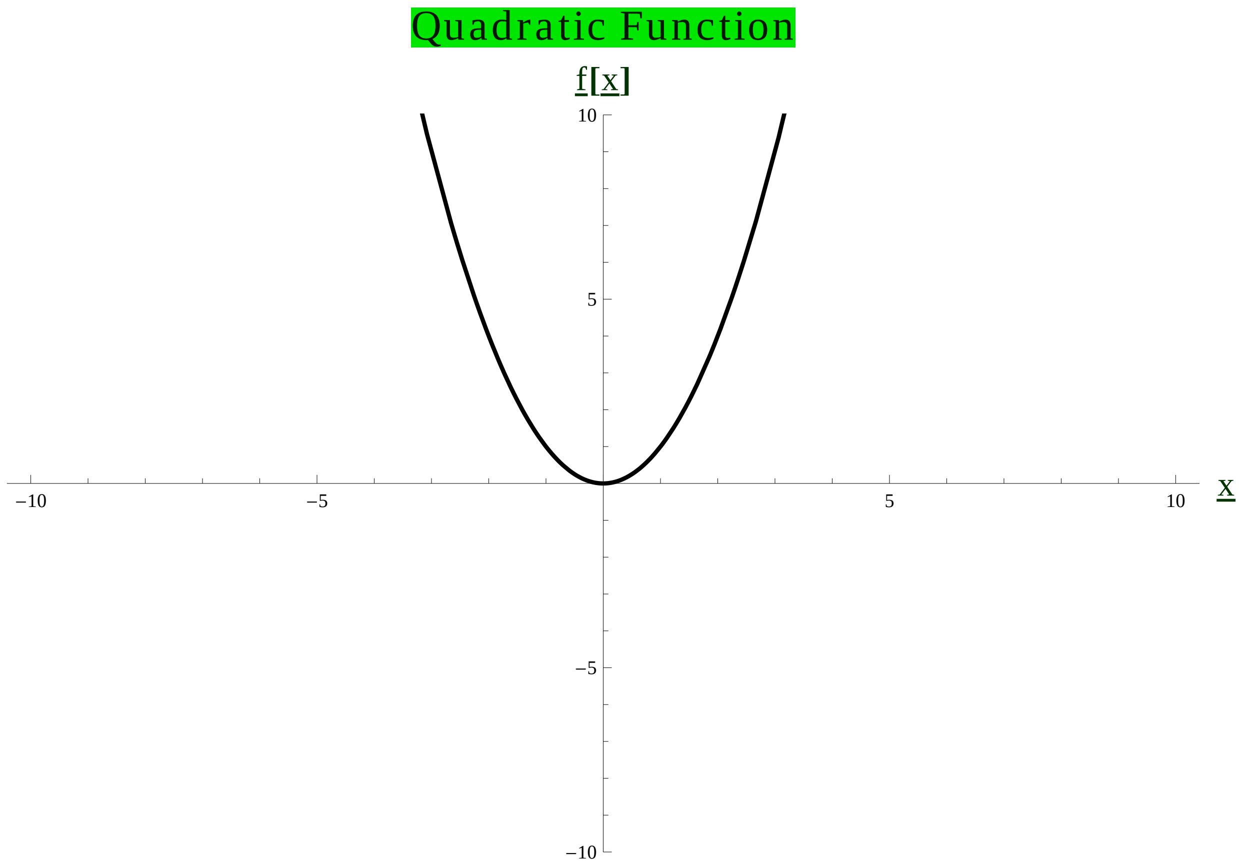
$$f[x] = f[-x]$$
$$x = -x$$
$$\text{False}$$

Testing odd parity:

$$-f[x] = f[-x]$$
$$-x = -x$$
$$\text{True}$$

Quadratic function $T[x] = x^2$

Graph



Domain

The domain of the function is ℝ because the graph is continuous.

Range

The range of the function is $[0, \infty)$ because the square of any number is always greater than 0.

Intervals of Increasing/Decreasing

The function is decreasing on the interval $(-\infty, 0]$ because the values of the ordinates decrease as the values of the abscissa increase.
The function is increasing on the interval $[0. \infty)$ because the values of the ordinates increase as the values of the abscissa increase.

Intervals of Concavity

The function is concave up for ℝ because it is decreasing at a decreasing rate on the interval $(-\infty, 0]$ and increasing at an increasing rate for the interval $[0. \infty)$.

Parity

Testing even parity:

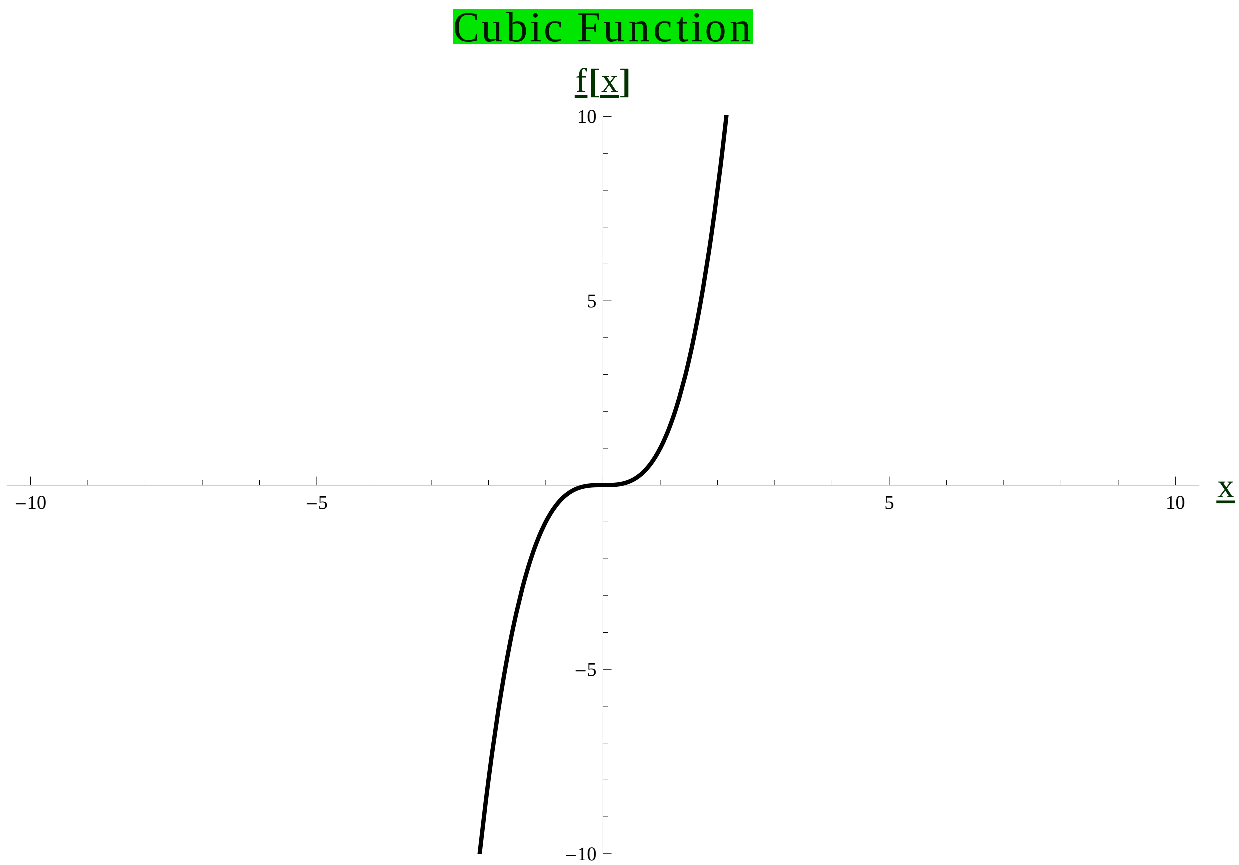
$$f[x] = f[-x]$$
$$x^2 = (-x)^2$$
$$x^2 = x^2$$
True

Testing odd parity:

$$-f[x] = f[-x]$$
$$-x^2 = (-x)^2$$
$$-x^2 = x^2$$
False

Cubic function $T[x] = x^3$

Graph



Domain

The domain of the function is ℝbecause the graph is continuous.

Range

The range of the function is ℝ because the graph is continuous and has a constant slope.

Intervals of Increasing/Decreasing

The function is increasing on the interval ℝ because the values of the ordinates increase as the values of the abscissa increase.

Intervals of Concavity

The function is concave up on the interval $[0, \infty)$ because it is increasing at an increasing rate.
The function is concave down on the interval $(-\infty, 0]$ because it is increasing at a decreasing rate.

Parity

Testing even parity:

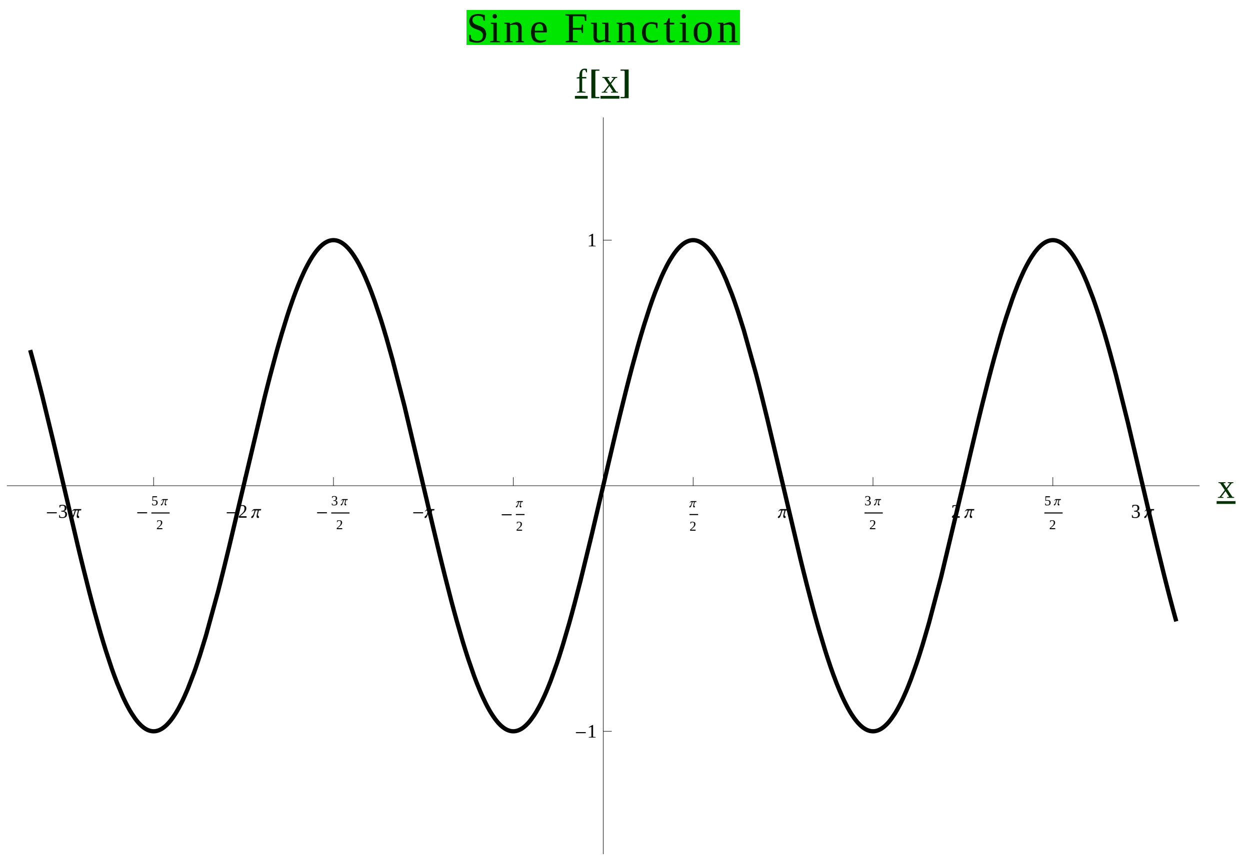
$$f[x] = f[-x]$$
$$x^3 = (-x)^3$$
$$x^3 = -x^3$$
False

Testing odd parity:

$$-f[x] = f[-x]$$
$$-x^3 = (-x)^3$$
$$-x^3 = -x^3$$
True

Sine function $T[x] = \text{Sin}[x]$

Graph



Domain

The domain of the function is ℝ because the graph is continuous.

Range

The range of the function is $[-1, 1]$. The function has a period of 2Π Thus, the maximum, 1 at $x = \frac{\pi}{2}$, and the minimum, -1 at $x = 3\,\frac{\pi}{2}$, repeat. Therefore, the min and max values of the range do not grow larger than 1 or lower than -1.

Intervals of Increasing / Decreasing

The function is decreasing on the intervals $\left[\frac{-\pi}{2} + 2\,\pi\,n,\ \frac{\pi}{2} + 2\,\pi\,n\right]$ where $n \in \mathbb{Z}$ because the values of the ordinates decrease as the values of the abscissa increase.
The function is increasing on the intervals $\left[\frac{\pi}{2} + 2\,\pi\,n,\ \frac{3\,\pi}{2} + 2\,\pi\,n\right]$ where $n \in \mathbb{Z}$ because the values of the ordinates increase as the values of the abscissa increase.

Intervals of Concavity

The function is concave up on the intervals $[-\pi + 2\,\pi\,n,\ 0 + 2\,\pi\,n]$ where $n \in \mathbb{Z}$ because the function is decreasing at a decreasing rate on the intervals $\left[-\pi + 2\,\pi\,n,\ \frac{-\pi}{2} + 2\,\pi\,n\right]$ where $n \in \mathbb{Z}$ and increasing at an increasing rate for the intervals $\left[\frac{-\pi}{2} + 2\,\pi\,n,\ 0 + 2\,\pi\,n\right]$where $n \in \mathbb{Z}$.
The function is concave down on the intervals $[0 + 2\,\pi\,n,\ \pi + 2\,\pi\,n]$ where $n \in \mathbb{Z}$ because the function is increasing at a decreasing rate on the intervals $\left[0 + 2\,\pi\,n,\ \frac{\pi}{2} + 2\,\pi\,n\right]$ where $n \in \mathbb{Z}$ and decreasing at an increasing rate for the intervals $\left[\frac{\pi}{2} + 2\,\pi\,n,\ \pi + 2\,\pi\,n\right]$where $n \in \mathbb{Z}$.

Parity

Testing even parity:

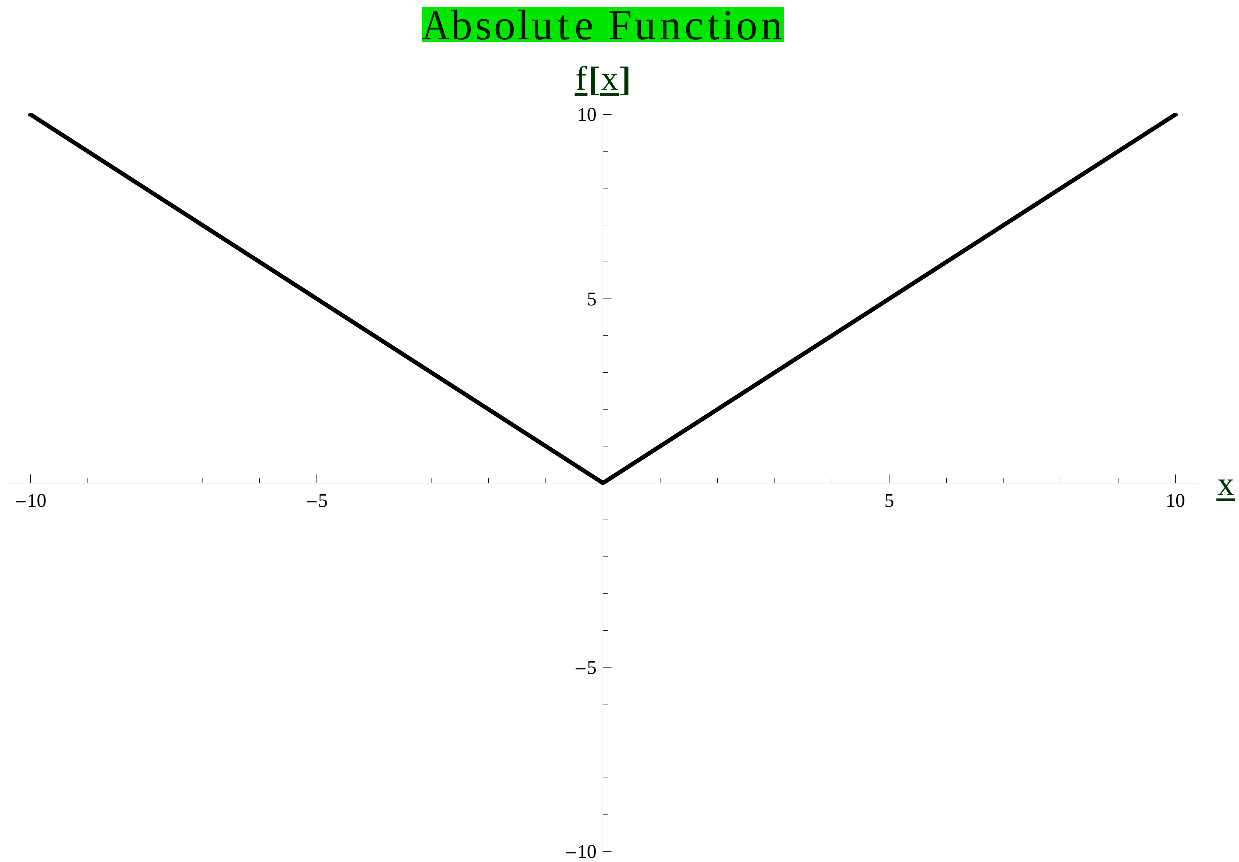
$$f[x] = f[-x]$$
$$\text{Sin}[x] = \text{Sin}[-x]$$
False

Testing odd parity:

$$-f[x] = f[-x]$$
$$-\text{Sin}[x] = \text{Sin}[-x]$$
True

Absolute Value function $T[x] = |x|$

Graph



Domain

The domain of the function is ℝ because the graph is continuous.

Range

The range of the function is $(0, \infty)$ because the absolute value of any number is always either positive or 0.

Intervals of Increasing/Decreasing

The function is decreasing on the interval $(-\infty, 0]$ because the values of the ordinates decrease as the values of the abscissa increase.
The function is increasing on the interval $[0. \infty)$ because the values of the ordinates increase as the values of the abscissa increase.

Intervals of Concavity

The function does not have any concavity on any interval because the function does not have any change in slope.

Parity

Testing even parity:

$$f[x] = f[-x]$$
$$\text{Abs}[x] = \text{Abs}[-x]$$
$$x = x$$
True

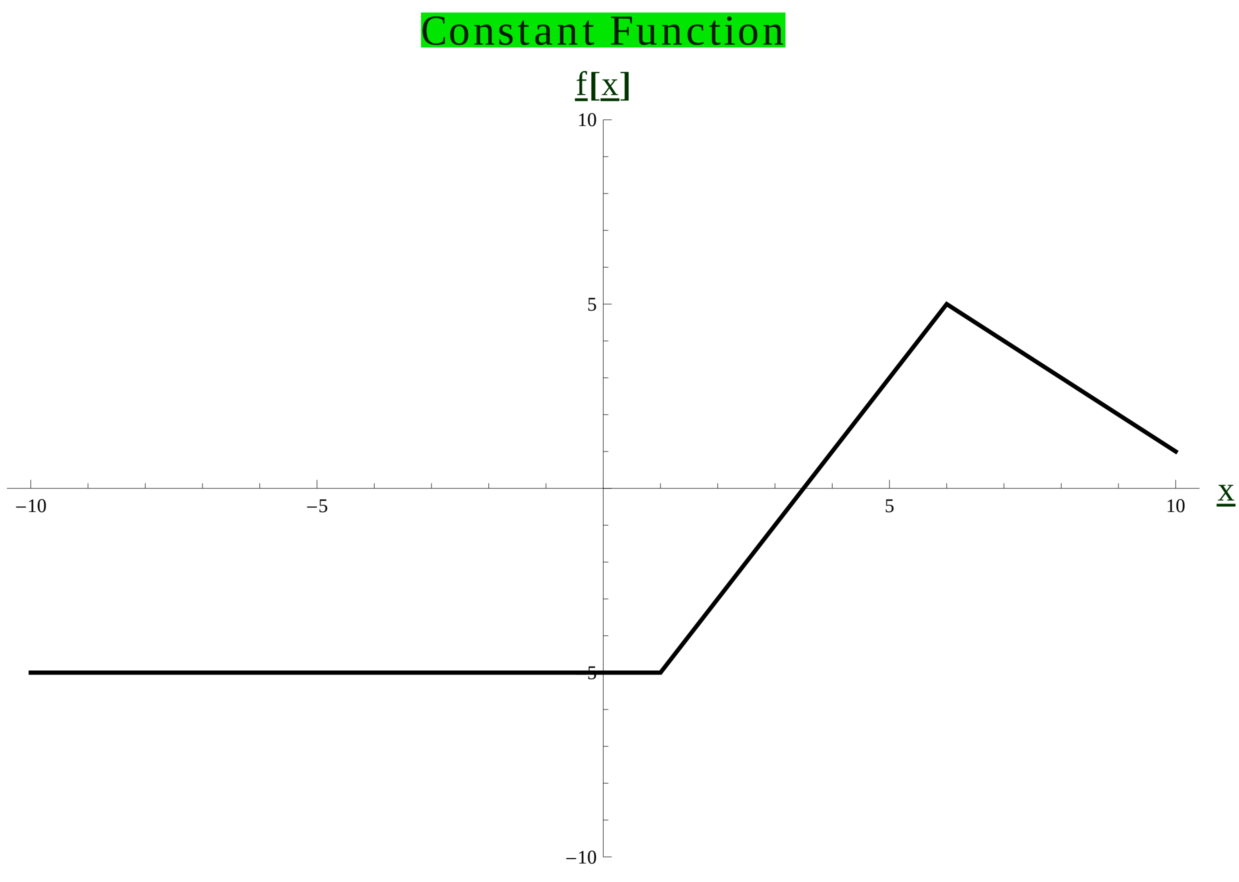
Testing odd parity:

$$-f[x] = f[-x]$$
$$-\text{Abs}[x] = \text{Abs}[-x]$$
$$-x = x$$
False

Piecewise-defined function

$$T[x] = \begin{cases} -5 & \text{for } x < 1 \\ -5 + 2(x-1) & \text{for } 1 \leq x < 6 \\ 5 + 2(x-6) - 3(x-6) & \text{for } 6 \leq x \leq \infty. \end{cases}$$

Graph



Domain

The domain of the function is ℝ because the graph is continuous.
However, all piecewise-defined functions do not have a domain of $(-\infty, \infty)$.

Range

The range of the function is [−5, 5] .
However, all piecewise-defined functions do not have a range of [-5, 5] .

Intervals of Increasing/Decreasing

The function is decreasing on the interval [6, ∞) because the values of the ordinates decrease as the values of the abscissa increase.
The function is increasing on the interval [1, 6] because the values of the ordinates increase as the values of the abscissa increase.

Intervals of Concavity

The function does not have any concavity on any interval because the function does not have any change in slope.

Parity

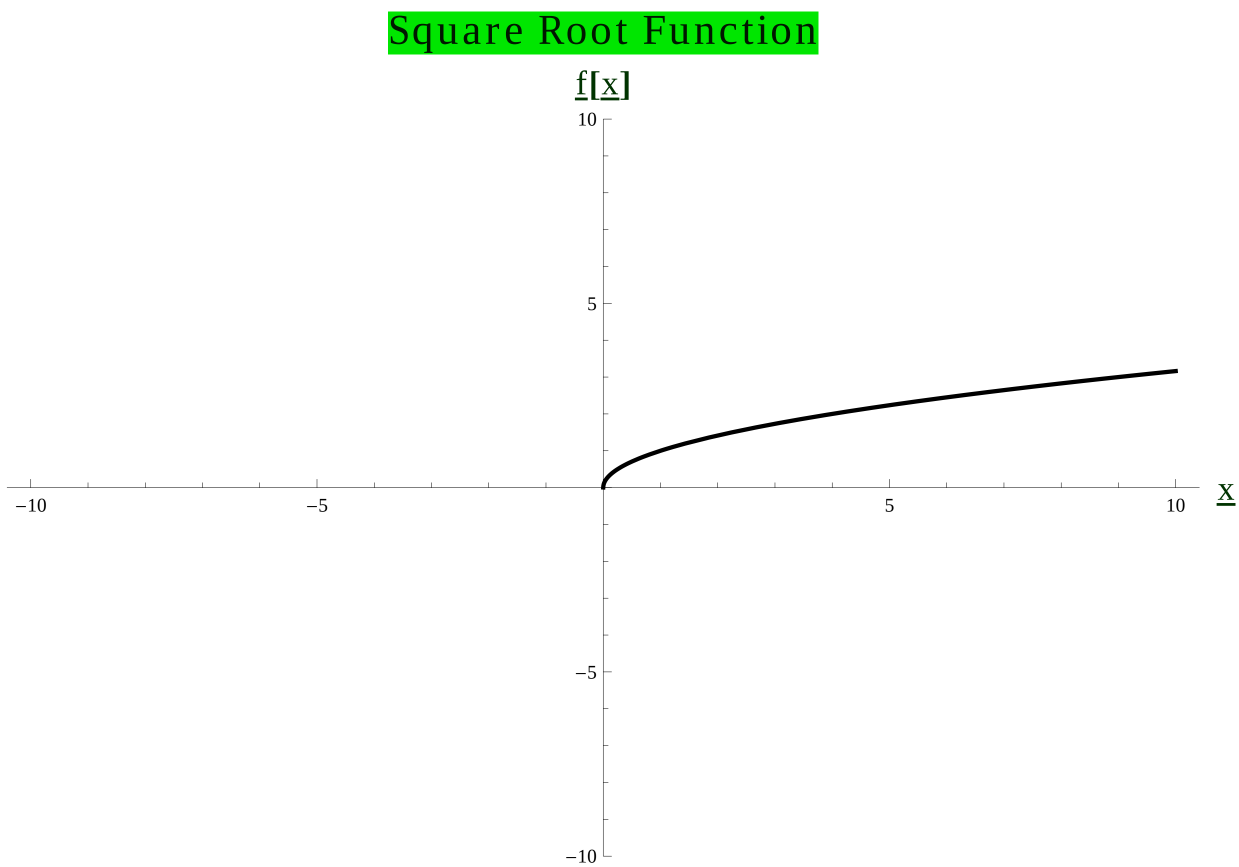
Even parity: This function is not even because a reflection across the vertical axis will not produce the same graph.

Testing odd parity: This function is not odd because a 180 degree rotation about the origin will not produce the same graph.

Some piecewise-defined functions can have even or odd symmetry.

Square Root function $T[x] = \sqrt{x}$

Graph



Domain

The domain of this function is (0, ∞) because an imaginary number is produced when $x < 0$.

Range

The range of this function is (0, ∞) because the square root of any non-negative number (see domain) is always non-negative.

Intervals of Increasing/Decreasing

The function is increasing on the interval [0. ∞) because the values of the ordinates increase as the values of the abscissa increase.

Intervals of Concavity

The function is concave up on its domain because it is increasing at an increasing rate.

Parity

Testing even parity:

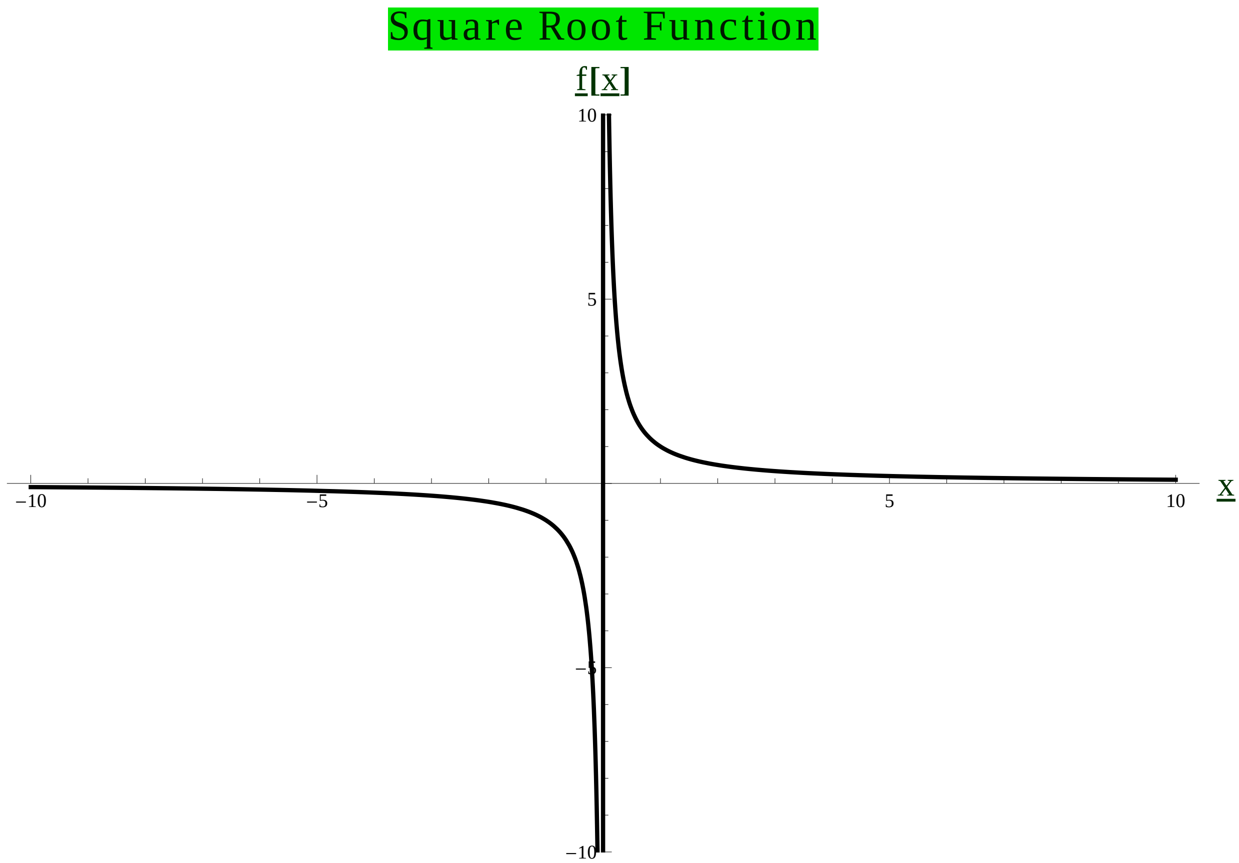
$$f[x] = f[-x]$$
$$\sqrt{x} = \sqrt{-x}$$
False

Testing odd parity:

$$-f[x] = f[-x]$$
$$-\sqrt{x} = \sqrt{-x}$$
False

Reciprocal function $T[x] = \frac{1}{x}$

Graph



Domain

The domain of this function is $(-\infty, 0) \cup (0, \infty)$ because there is an asymptote at $x = 0$.

Range

The range of this function is $(-\infty, 0) \cup (0, \infty)$ because there is an asymptote at $y = 0$.

Intervals of Increasing/Decreasing

The function is decreasing on the interval $(-\infty, 0) \cup (0, \infty)$ because the values of the ordinates decrease as the values of the abscissa increase. There is discontinuity at $x = 0$; therefore, the function cannot increase or decrease in an range including this point.

Intervals of Concavity

The function is concave up on the interval $[0, \infty)$ because it is decreasing at a decreasing rate.
The function is concave down on the interval $(-\infty, 0]$ because it is increasing at a decreasing rate.

Parity

Testing even parity:

$$f[x] = f[-x]$$
$$\frac{1}{x} = \frac{-1}{x}$$
False

Testing odd parity:

$$-f[x] = f[-x]$$
$$\frac{-1}{x} = \frac{-1}{x}$$
True

Exponential function $T[x] = b^x$

Graph

Domain

The domain of this function is ℝ because the graph is continuous.

Range

For ℝ, $x \neq 1$.

The range of this function is $(0, \infty)$ because any power of a non-negative number is always non-negative.

For $b = 1$

The range of this function is $[1, 1]$ because any power of 1 is 1.

Intervals of Increasing/Decreasing

For $b = 1$;

The function is not increasing or decreasing on any interval because the function does not have any slope.

For $b > 1$;

The function is increasing on ℝ because the values of the ordinates increase as the values of the abscissa increase.

For $b < 1$;

The function is decreasing on ℝ because the values of the ordinates decrease as the values of the abscissa increase.

Intervals of Concavity

For $b = 1$;

The function does not have any concavity on any interval because the function does not have any slope.

For $b > 1$;

The function is concave up on its domain because it is increasing at an increasing rate.

For $b < 1$;

The function is concave up on its domain because it is decreasing at a decreasing rate.

Parity

***b* = 1**

Testing even parity:

$$f[x] = f[-x]$$
$$1^x = 1^{(-x)}$$
$$1 = 1$$
$$\text{True}$$

Testing odd parity:

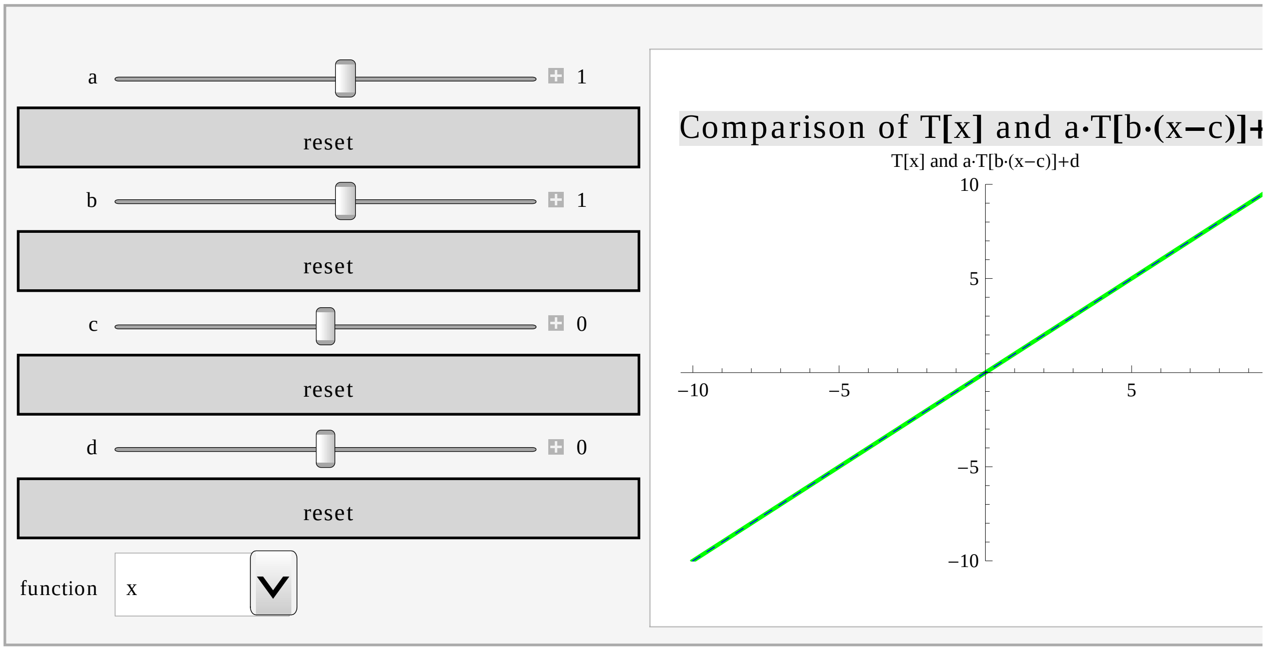
$$-f[x] = f[-x]$$
$$-(1^x) = 1^{(-x)}$$
$$-1 = 1$$
$$\text{False}$$

***b* ≠ 1**

For all other values of b, a reflection across the vertical axis or a 180 degree rotation about the origin will not produce the same graph; therefore, it does not have even or odd symmetry.

Transformations

Dynamic

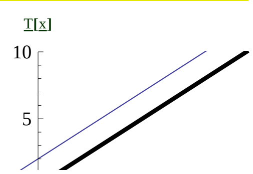


Static

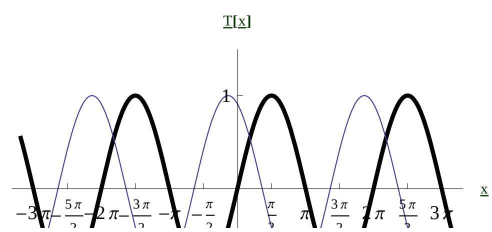
Translations

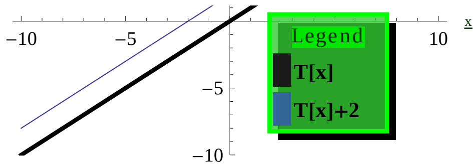
Vertical (left)and horizontal (right) translations

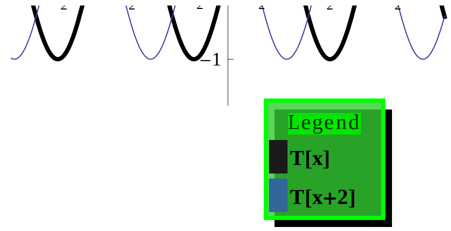
Linear function vertical translation



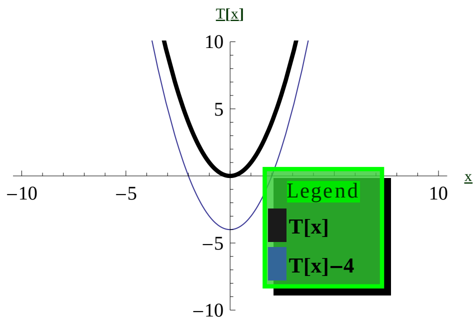
Sine function horizontal translation



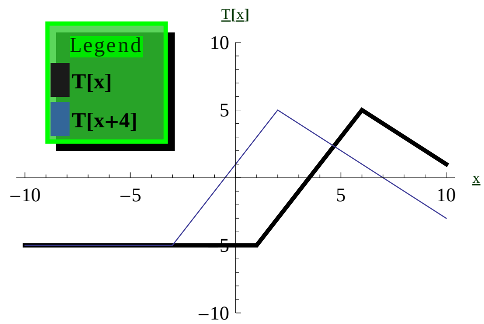




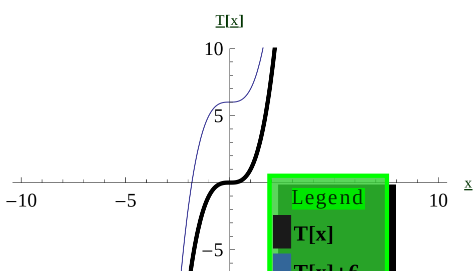
quadratic function vertical translation



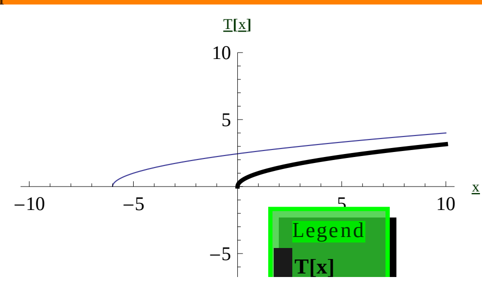
Piecewise function horizontal translation



Cubic function vertical translation



Square Root function horizontal translation



Translation

A translation is the change of a function where all abscissa or all ordinates are added to or subtracted by the same value.

Translations can be categorized into veritcal (yellow above) and horozontal (orange above)

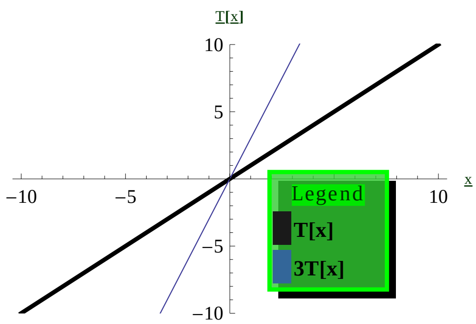
As the value of c increases, all abscissa of the function are increased by the value of c; as the value of c decreases, all abscissa of the function are decreased by the value of c.

As the value of d increases, all ordinates of the function decrease by the value of d; as the value of d decreases, all ordinates of the function are increased by the value of d.

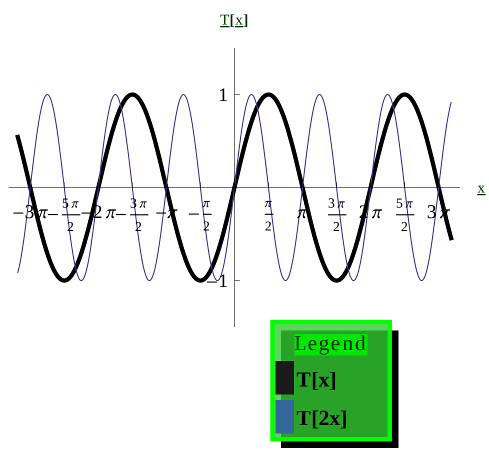
Dilation

Vertical (left)and horizontal (right) dilations

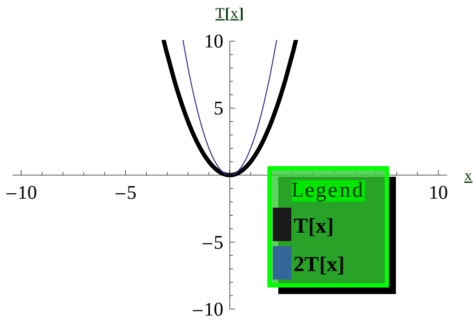
Linear function vertical dilation



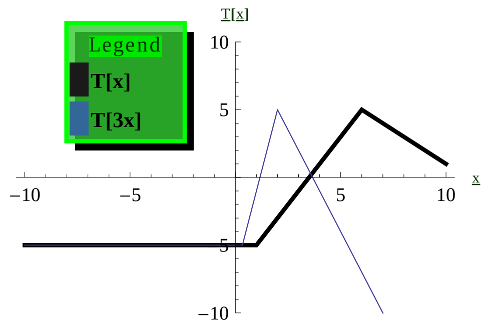
Sine function horozontal dilation



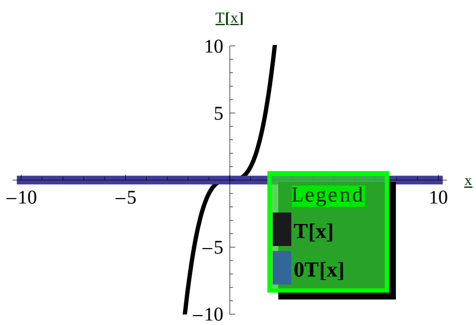
quadratic function vertical dilation



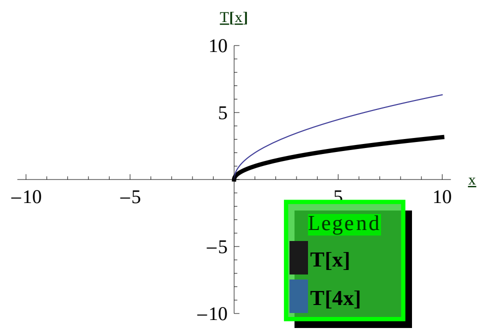
Piecewise function horizontal dilation



Cubic function vertical dilation



Square Root function horizontal dilation



Dilations

A dilation is the change of a function where all abscissa or all ordinates are multiplied by or divided by the same value.

Each dilation can be classified as vertical (yellow above) or horozontal (orange above) and stretch or compression.

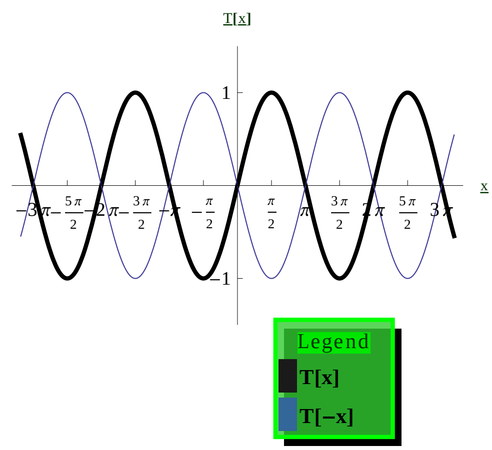
When the value of b is in the inverval (0, 1), all abscissa are increased by a factor of $\frac{1}{b}$; when the value of b is in the interval (1,∞), all abscissa are decreased by a factor of $\frac{1}{b}$; when the value of b is 0, a constant function of $y = 0$ is produced; when the value of b is 1, the abscissa values remain unchanged.

When the value of a is in the inverval (0, 1), all abscissa are decreased by a factor of a; when the value of a is in the interval (1,∞), all abscissa are increased by a factor of a; when the value of a is 0, a constant function of $y = 0$ is produced; when the value of a is 1, the ordinate values remain unchanged.

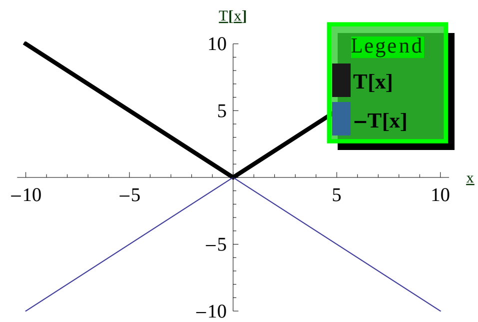
Reflections

Vertical (left)and horizontal (right) reflections

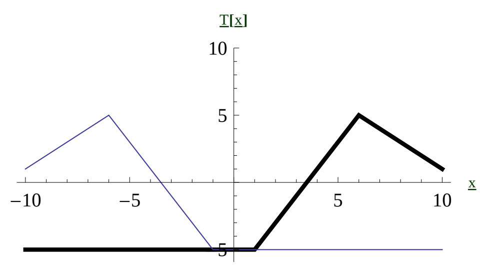
Sine function horizontal reflection



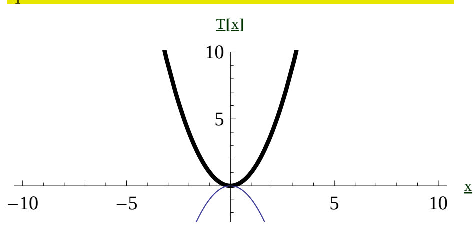
Absolute value function vertical reflection

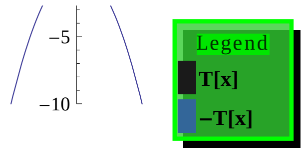


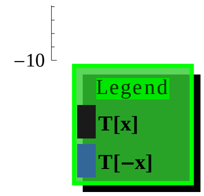
Piecewise function horizontal reflection



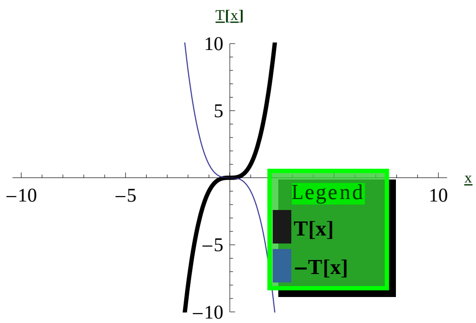
quadratic function vertical reflection



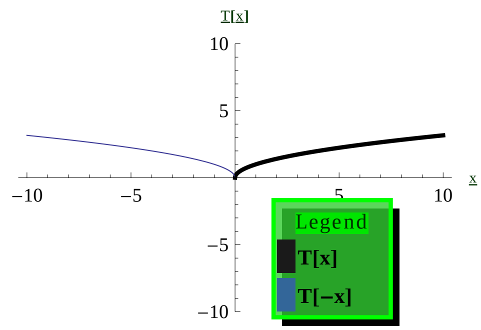




Cubic function vertical reflection



Square Root function horizontal reflection



Reflections

A reflection is the change of a function where all abscissa or all ordinates are multiplied by -1.

A vertical reflection occurs when a is negative. When a is negative, all ordinates take their opposite signs.

A Horizontal reflection occurs when b is negative. When b is negative, all abscissa take their opposite signs.

When the value of b is negative, the function is reflected across the line $x = x + c$.

When the value of a is negative, the function is reflected across the line $y = d$.

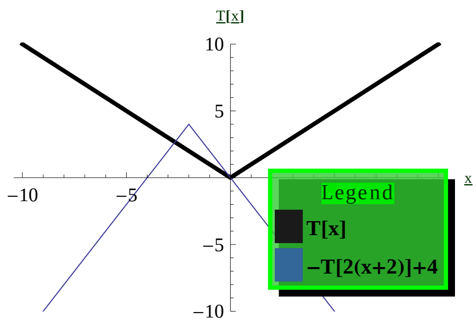
Combinations of Transformations

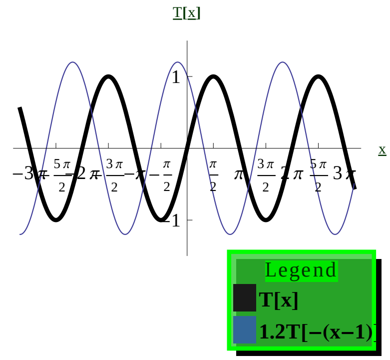
Composition of Transformations

Absolute value transformation

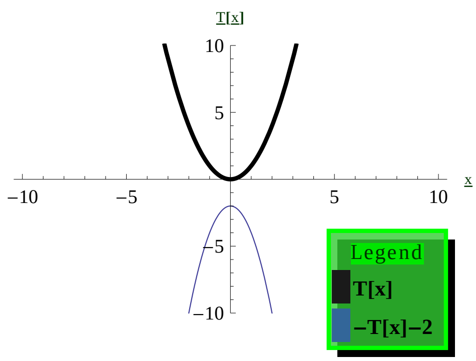
Sine function transformation

Absolute value transformation

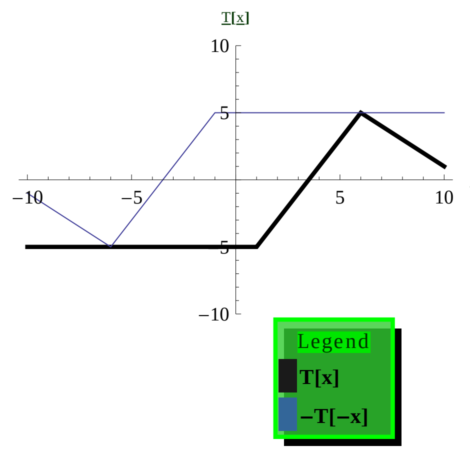




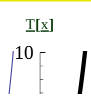
quadratic function transformation



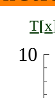
Piecewise function transformation

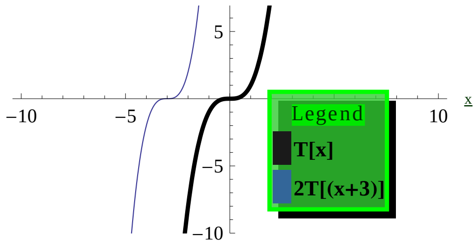


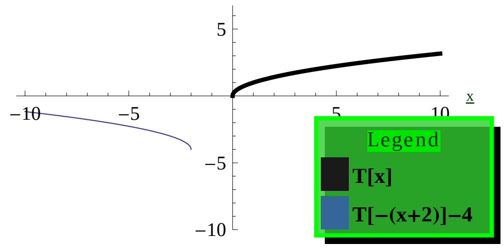
Cubic function transformation



Square Root function transformation







Compositions

A composition is a change of a function where the abscissa and/or the ordinates are multiplied by, divided by, added to, and/or subtracted by the same respective values.

To graph a combination of functions, one should take a point from the original function, multiply the abscissa by a, subtract c from the abscissa, multiply the ordinate by b, and then add d to the ordinate. If the value of a or b is negative, the reflection should be performed first so that one only has to reflect across the vertical and/or horizontal axis (opposed to reflecting across $y = d$ and $x = x + c$ respectively). It does not matter if one changes the value of the ordinate before the abscissa, or visa versa.