On the Sum-Rate Capacity of Non-Symmetric Poisson Multiple Access Channel

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Abstract—In this paper, we characterize the sum-rate capacity of the non-symmetric Poisson multiple access channel (MAC). While the sum-rate capacity of the symmetric Poisson MAC has been characterized in the literature, the special property exploited in the existing study for the symmetric case does not hold for the non-symmetric channel anymore. Hence, the method used in the symmetric case does not work for the non-symmetric case. We obtain the optimal input that achieves the sum-rate capacity by solving a non-convex optimization problem. We show that, for certain channel parameters, it is optimal for a single user to transmit in order to achieve the sum-rate capacity. This is in sharp contrast to the Gaussian MAC, in which it is optimal for all users to transmit simultaneously in order to achieve the sum-rate capacity.

Index Terms—Poisson channels, multi-access channels, optimal power allocation, sum-rate.

I. INTRODUCTION

The Poisson channel is a suitable model for optical communications when the receivers employ photon-sensitive devices to record the arrival of photons. The point-to-point Poisson channel has been well understood under various scenarios such as single antenna [1], multiple antenna [2] and fading [3], [4]. However, multiuser Poisson channels are less well understood, with few exceptions on the multiple-access channel (MAC) [5], on the broadcast channel [6] and on the Poisson interference channel [7].

Of particular relevance to our study is [5] that thoroughly investigated the Poisson MAC. In particular, [5] showed that approximating the complicated continuous time continuous input discrete output Poisson MAC with a simpler discrete time binary input binary output MAC does not bring any loss in terms of the channel capacity region. Using this result, [5] then characterized the sum-rate capacity of the symmetric MAC, in which the channel gains and power constraints of all users are the same, under the maximum power constraint. [5] further characterized the boundary points on the capacity region of the symmetric Poisson MAC under the maximum power constraint and the sum capacity under the average power constraint.

In this paper, we focus on characterizing the sum-rate capacity of the non-symmetric Poisson MAC, in which the channel gains and power constraints at different users are not necessarily the same. This scenario naturally arises in multiuser optical communications when the transmitters have different distances to the receiver or have different transmitting power. Unfortunately, the method used in [5] to characterize the sum-rate capacity for the symmetric case does not apply for the non-symmetric case anymore. In particular, the method in [5] relies on the property that the objective function involved is a Schur concave function for the symmetric Poisson MAC, which greatly simplifies the analysis. However, in the non-symmetric channel, the objective function is not symmetric, and hence the objective function is not Schur concave anymore.

As the result, we have to resort to a different approach than the one in [5]. Towards this goal, we show that characterizing the sum-rate capacity is equivalent to solving a non-convex optimization problem. We show that there are at most four possible candidates for the optimal solution to this optimization problem. Two of these four possible candidates correspond to the cases when only one user transmits. We further show that, for some channel parameters, it is indeed optimal to allow only one user to transmit in order to achieve the sum-rate capacity under the maximum power constraint. This is in sharp contrast to the Gaussian MAC with an average power constraint, in which it is always optimal for all users to transmit simultaneously to achieve the sum-rate capacity. We also identify conditions under which it is optimal for both users to transmit in order to achieve the sum-rate capacity.

The remainder of the paper is organized as follows. In Section II, we introduce the system model. In Section III, we present our main result. In Section IV, we further investigate the solution obtained and identify conditions under which it is optimal to allow either one or two users to transmit. In Section V, we specialize our results to the symmetric case and show that our results recover that of [5]. We present several numerical examples in Section VI. Finally, we provide concluding remarks in Section VII.

II. SYSTEM MODEL

In this section, we introduce the model considered in this paper. As shown in the Fig. 1, we consider Poisson MAC with $N$ users communicating with a single receiver. Let $X_n(t)$ be the input of the $n^{th}$ user and $Y(t)$ be the doubly-stochastic
Poisson process observed at the receiver. The relationship between them can be described as:

\[
Y(t) = P\left(\sum_{n=1}^{N} S_n X_n(t) + \lambda \right),
\]

in which \(S_n\) is the channel response between the \(n\)th user and the receiver, \(\lambda\) is the dark current at the receiver, and \(P(\cdot)\) is the non-linear transformation converting the light strength to the doubly-stochastic Poisson process that records the timing and number of photon’s arrivals. In particular, for any time interval \([t, t+\tau]\), the probability that there are \(j\) photons arriving at the receiver is

\[
\Pr\{Y(t+\tau) - Y(t) = j\} = \frac{e^{-\Lambda} \Lambda^j}{j!},
\]

where \(\Lambda = \int_t^{t+\tau} \left[ \sum_{n=1}^{N} S_n X_n(t') + \lambda \right] dt'.
\]

We consider the peak power constraint, i.e., the transmitted signal \(X_n(t)\) must satisfy the following constraint:

\[
0 \leq X_n(t) \leq A_n,
\]

where \(A_n\) is the maximum power allowed for the user \(n\).

The channel is called symmetric when \(S_i A_i = S_j A_j\), \(\forall i, j \in [1, N]\) and is non-symmetric when \(S_i A_i \neq S_j A_j\) for some \(i \neq j\). We are interested in characterizing the sum-rate capacity of the Poisson MAC. The sum-rate capacity for the symmetric case has been characterized in [5] by exploiting a special property possessed only by the symmetric channel. We focus on the non-symmetric case in this paper.

III. OPTIMALITY CONDITIONS

In this section, we focus on the two-user case, i.e., \(N = 2\). The approach developed here can be generalized to the general value of \(N\). It has been shown in [5] that the continuous input, discrete output Poisson MAC channel can be converted to a much simpler binary input, binary output MAC channel. In particular, the input waveform can be limited to be piecewise constant waveforms with two levels 0 or \(A_n\) for the \(n\)th transmitter. Let \(p_n\) be the duty cycle of the \(n\)th transmitter, it has been shown in [5] that the sum-rate capacity is given by

\[
C_{\text{sum}}(p_1, p_2) = \max_{p_1, p_2} I_{X_1, X_2; Y}(p_1, p_2)
\]

\[
0 \leq p_1, p_2 \leq 1,
\]

in which

\[
I_{X_1, X_2; Y}(p_1, p_2) = (1 - p_1)(1 - p_2)\lambda \log \lambda
\]

\[
+ p_1(1 - p_2)(S_1 A_1 + \lambda) \log(S_1 A_1 + \lambda)
\]

\[
+ (1 - p_1)p_2(S_2 A_2 + \lambda) \log(S_2 A_2 + \lambda)
\]

\[
+ p_1 p_2(S_1 A_1 + S_2 A_2 + \lambda) \log(S_1 A_1 + S_2 A_2 + \lambda)
\]

\[
- (S_1 A_1 p_1 + S_2 A_2 p_2 + \lambda) \log(S_1 A_1 p_1 + S_2 A_2 p_2 + \lambda).
\]

For notational convenience, we let \(b \triangleq S_1 A_1 + \lambda, c \triangleq S_2 A_2 + \lambda, d \triangleq S_1 A_1 + S_2 A_2 + \lambda, A \triangleq \lambda \log \lambda, B \triangleq b \log(b), c \triangleq c \log(c), D \triangleq d \log(d).\)

The optimization problem (5) has been solved by [5] for the symmetric case \(S_1 A_1 = S_2 A_2\). In particular, [5] showed that the objective function \(I_{X_1, X_2; Y}(p_1, p_2)\) is a Schur concave function when \(S_1 A_1 = S_2 A_2\). As the result, if \((\hat{p}_1, \hat{p}_2)\) is the optimal solution to (5) for the symmetric case, \(\hat{p}_1\) must be equal to \(\hat{p}_2\). Hence, the problem can be converted into a one dimension optimization problem, which can be solved easily.

However, the situation for the non-symmetric case is different. In particular, when \(S_1 A_1 \neq S_2 A_2, I_{X_1, X_2; Y}(p_1, p_2)\) is not a Schur concave function anymore. This can be observed from the fact that a Schur concave function must be a symmetric function when \(S_1 A_1 \neq S_2 A_2\). Therefore the techniques developed in [5] for the symmetric case cannot be extended to the non-symmetric case. Furthermore, for general values of \(S_n A_n\) and \(\lambda, I_{X_1, X_2; Y}(p_1, p_2)\) is not a symmetric function when \(S_1 A_1 \neq S_2 A_2\). Therefore the techniques developed in [5] for the symmetric case cannot be extended to the non-symmetric case. In the following, we solve this non-concave optimization problem. We start with the necessary KKT conditions (as the problem is not convex, hence these conditions are not sufficient conditions). For convenience, we write \(I_{X_1, X_2; Y} = I\) and hence the corresponding Lagrangian equation is:

\[
\mathcal{L} = -I + \eta_1(p_1 - 1) - \eta_2 p_1 + \eta_3(p_2 - 1) - \eta_4 p_2.
\]

The optimal solution \((\hat{p}_1, \hat{p}_2)\) must satisfy the following KKT constraints:

\[
\frac{\partial I}{\partial p_1}(\hat{p}_1, \hat{p}_2) - \eta_1 + \eta_2 = 0,
\]

\[
\frac{\partial I}{\partial p_2}(\hat{p}_1, \hat{p}_2) - \eta_3 + \eta_4 = 0,
\]

\[
\eta_1(\hat{p}_1 - 1) = 0,
\]

\[
\eta_2\hat{p}_1 = 0,
\]

\[
\eta_3(\hat{p}_2 - 1) = 0,
\]

\[
\eta_4\hat{p}_2 = 0,
\]

where

\[
\frac{\partial I}{\partial p_1} = -(1 - p_2)A + (1 - p_2)B - p_2 C + p_2 D
\]

\[
- S_1 A_1 \log(S_1 A_1 p_1 + S_2 A_2 p_2 + \lambda) - S_1 A_1,
\]

and

\[
\frac{\partial I}{\partial p_2} = -(1 - p_1)A - p_1 B + (1 - p_1)C + p_1 D
\]

\[
- S_2 A_2 \log(S_1 A_1 p_1 + S_2 A_2 p_2 + \lambda) - S_2 A_2.
\]
In order to solve these KKT conditions, we have 16 different cases corresponding to different combinations of active constraints (i.e., either \( \eta_i = 0 \) or not for \( i = 1, \ldots, 4 \)). For example, when \( \eta_1 = 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 = 0 \), KKT conditions above can be simplified to
\[
\frac{\partial I}{\partial p_1} (p_1, p_2) = 0, \\
\frac{\partial I}{\partial q} (p_1, p_2) - \eta_3 = 0, \\
\eta_3 (p_2 - 1) = 0,
\]
from which we obtain
\[
\hat{p}_1 = \frac{1}{S_1 A_1} \left( \frac{d^d}{c(S_2 A_2 + \lambda)} \right)^{\frac{1}{\lambda-1}} e^{-1} - \frac{c}{S_1 A_1}, \\
\hat{p}_2 = 1.
\]
As \( I(\hat{p}_1, 0) > I(\hat{p}_1, 1) \), (10) is clearly not an optimal solution.

Using similar arguments, we can show that among these 16 possible combinations, 13 constraint combinations result in non-optimal solutions. We are left with the following three possible scenarios:

**Scenario 1:** \( \eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 = 0 \): The KKT conditions are simplified to
\[
\frac{\partial I}{\partial p_1} (p_1, p_2) = 0, \\
\frac{\partial I}{\partial p_2} (p_1, p_2) = 0.
\]
From (8) and (9), we can see that both \( \frac{\partial I}{\partial p_1} \) and \( \frac{\partial I}{\partial p_2} \) are nonlinear functions of \((p_1, p_2)\). Hence, there might be multiple \((p_1, p_2)\) pairs satisfying (11) and (12) simultaneously. However, we now show that there are at most 2 possible \((p_1, p_2)\) pairs that satisfy (11) and (12) simultaneously.

First, by (11)\( S_2 A_2 - (12) \times S_1 A_1 \), we have
\[
S_2 A_2 (-(1-p_2) A + (1-p_2) B - p_2 C + p_2 D) = S_1 A_1 (-(1-p_1) A - p_1 B - (1-p_1) C + p_1 D).
\]
Using (13), we can write \( p_2 \) in terms of \( p_1 \):
\[
p_2 = \frac{-B + A - \frac{S_1 A_1}{S_2 A_2} A + \frac{S_1 A_1}{S_2 A_2} C}{A - B + C + D} + \frac{S_1 A_1}{S_2 A_2} p_1.
\]

It is clear that \( f(p_1) \) is a linear function of \( p_1 \).

Using \( \frac{\partial I}{\partial p_2} = 0 \), we can write \( p_2 \) in terms of \( p_1 \):
\[
p_2 = \frac{1}{S_2 A_2} \left( \exp \left( \frac{1}{S_2 A_2} \left( -(1-p_1) A - p_1 B + (1-p_1) C + p_1 D - S_2 A_2 \right) \right) - \frac{S_1 A_1 p_1 + \lambda}{S_2 A_2} \right).
\]

Similarly, it is easy to check that \( g''(p_1) > 0 \), and hence \( g(p_1) \) is a strictly convex function of \( p_1 \).

We have just converted (11) and (12) into equivalent forms:
\[
p_2 = f(p_1),
\]
\[
p_2 = g(p_1).
\]

\((p_1, p_2)\) pairs where \( f(p_1) \) and \( g(p_1) \) intersect with each other will satisfy (11) and (12) simultaneously. As \( f(p_1) \) is a linear function of \( p_1 \), while \( g(p_1) \) is a strictly convex function of \( p_1 \), they can have at most two intersecting points as shown in Fig. 2.

Therefore, there could be at most two pairs of \((p_1, p_2)\) that can satisfy both conditions simultaneously. Let these solutions be \((\hat{p}_1, \hat{p}_2)\) and \((\tilde{p}_1, \tilde{p}_2)\). We then need to check whether \((\hat{p}_1, \hat{p}_2)\) is in \([0, 1] \times [0, 1]\) or not. If yes, we keep it. If not, then for the presentation convenience, we replace it with \((0, 0)\). We do the same for \((\tilde{p}_1, \tilde{p}_2)\).

**Scenario 2:** \( \eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 \neq 0 \):

Solving the corresponding KKT conditions, we can easily obtain
\[
\tilde{p}_1 = \frac{1}{S_1 A_1} \left( \frac{b^b}{\lambda^b} \right)^{\frac{1}{\lambda-1}} e^{-1} - \frac{\lambda}{S_1 A_1}, \\
\tilde{p}_2 = 0.
\]

It is easy to check that \( 0 < \tilde{p}_1 < 1 \), and hence \((\tilde{p}_1, 0)\) is a valid input.

**Scenario 3:** \( \eta_1 = 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 = 0 \):

Solving the corresponding KKT conditions, we can easily obtain
\[
\tilde{p}_1 = 0, \\
\tilde{p}_2 = \frac{1}{S_2 A_2} \left( \frac{e^c}{\lambda^c} \right)^{\frac{1}{\lambda-1}} e^{-1} - \frac{\lambda}{S_2 A_2}.
\]

Similarly, it is easy to check that \( 0 < \tilde{p}_2 < 1 \), and hence \((0, \tilde{p}_2)\) is a valid input.

In summary, we have the following proposition.
Proposition 1. The optimal value \((\hat{p}_1, \hat{p}_2)\) that achieves the sum-rate capacity for the general Poisson MAC is

\[
(\hat{p}_1, \hat{p}_2) = \begin{cases} 
(0, p^*_2) & \text{if } I(0, p^*_2) \geq 
\max (I(\hat{p}_1, 0), I(\hat{p}_1, \hat{p}_2), I(\hat{p}_1', \hat{p}_2'))
\end{cases}
\]

and

\[
\lim_{S_2A_2 \to \infty} g(p_1) = \lim_{S_2A_2 \to \infty} g(0) = \lim_{S_2A_2 \to \infty} g(1) = \frac{1}{e},
\]

therefore, (16) and (17) do not intersect as \(S_2A_2 \to \infty\).

Proof. The proof is provided in Appendix B.

![Fig. 3: f(p1) and g(p1) have no intersection in 0 ≤ p1 ≤ 1 and 0 ≤ p2 ≤ 1, when S1A1 = 5, S2A2 = 50, and λ = 0.5.](image)

On the other hand, there are scenarios under which it is optimal for both users to transmit. For example, when \(S_1A_1 = 10, S_2A_2 = 15, \lambda = 0.5\), it is easy to check that it is optimal for both users to transmit in order to achieve the sum-rate capacity.

Motivated by these observations, in this section, we analyze (20) further to characterize conditions under which it is optimal to either allow one user or two users to transmit.

A. Optimality of Single User Transmission

In this subsection, we present conditions under which it is optimal for a single user to transmit. Intuitively speaking, we show that if one of the \(S_iA_i\)s is very large, then it is optimal for one user to transmit. We prove this by showing that (16) and (17) do not have an intersection in the desired region \([0, 1] \times [0, 1]\) when one \(S_iA_i\) is very large.

As the roles of users are symmetric, we restrict our analysis to \(S_2A_2 \to \infty\) in this section. We will show that as \(S_2A_2 \to \infty\), \(f(p_1)\) and \(g(p_1)\) do not intersect.

Lemma 2. We have

\[
\lim_{S_2A_2 \to \infty} f(p_1) = \lim_{S_2A_2 \to \infty} f(0) = \lim_{S_2A_2 \to \infty} f(1) = 1
\]

![Fig. 4: p2 vs. p1 as S2A2 → ∞](image)
Step 1: Among the four possible solutions in (20), we first rule out \((0, p_2^*)\) and \((p_1, 0)\). It is easy to check that, when \(S_1A_1 = S_2A_2\), \(\gamma_1 = 0\) and \(\gamma_2 = 0\). Hence, as discussed in Section IV-B, \((0, p_2^*)\) is not optimal, as we clearly have \(p_2^* > \gamma_2 = 0\). Similarly, \((p_1, 0)\) is not optimal, as \(p_1 > \gamma_1 = 0\). Hence, scenario 2 and scenario 3 cannot be optimal, and we are left with only scenario 1.

Step 2: We show that, if \((p_1, p_2)\) is a solution to (16) and (17) of scenario 1, then \(p_1\) must be equal to \(p_2\). This can be easily seen by setting \(S_1A_1 = S_2A_2\) in (14), which leads to \(p_1 = p_2\).

Step 3: We show that there is a unique pair \((p_1, p_2)\) that satisfies (16) and (17) of scenario 1. To prove the uniqueness of the solution, as illustrated in Fig. 5, we will show that \(g(0) > 0 = f(0)\) and \(g(1) < 1 = f(1)\). As \(g(\cdot)\) is a concave function while \(p_2 = p_1\) is a linear function, this proves that \(f(p_1)\) and \(g(p_1)\) have a single intersecting point in the range \(0 \leq p_1 \leq 1\).

Lemma 3. When \(S_1A_1 = S_2A_2\), we have \(g(1) < 1\) and \(g(0) > 0\).

Proof. Please see Appendix C for proof.

Hence it can be concluded that if \(S_1A_1 = S_2A_2\), then there is a unique solution to the problem and optimality \(\hat{p}_1 = \hat{p}_2\). This result is consistent with the one shown in [5].

VI. NUMERICAL EXAMPLES

In this section we use several numerical examples to illustrate results obtained in the previous sections.

Fig. 6: Optimal operating schemes vs \(S_1A_1\) and \(S_2A_2\)

Fig. 6 shows the optimal operating scenarios for different combinations of \(S_1A_1\) and \(S_2A_2\) when they range from 0 to 25. In generating this figure, we set \(\lambda = 0.5\). In Region-I, it is optimal for user 2 to transmit alone. Region-II correspond to the case it is optimal for both users to transmit. In Region-III, it is optimal for user 1 to transmit alone.

Fig. 7 shows the effect of increasing \(S_2A_2\) on the optimal value of \((\hat{p}_1, \hat{p}_2)\) when \(S_1A_1\) is constant. In this figure, \(S_1A_1 = 12.5\). We can see that when \(S_2A_2\) is small, the optimal value of \(\hat{p}_2 = 0\), i.e., it is optimal for user 2 to stay silent. We can also observe that once \(S_2A_2\) starts increasing and has noticeable value compared to \(S_1A_1\), \(\hat{p}_1\) starts decreasing while \(\hat{p}_2\) starts increasing, \(p_1\) and \(p_2\) intersect with each other, i.e. \(\hat{p}_1 = \hat{p}_2\), when \(S_1A_1 = S_2A_2\).

VII. CONCLUSION

In this paper, we have characterized the sum-rate capacity of the non-symmetric Poisson MAC. We have solved a non-convex optimization problem and have shown that, to achieve the sum-rate capacity, it is optimal to allow only one user to transit for certain channel parameters.

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Appendix A
Concavity of $I_{X_1,X_2;Y}(p_1,p_2)$

In this Appendix, we show that $I_{X_1,X_2;Y}(p_1,p_2)$ is not necessarily concave for general values of $S_1A_1, S_2A_2$ and $\lambda$. For $I_{X_1,X_2;Y}(p_1,p_2)$ to be concave, $\nabla^2 I$ needs to be semi-negative definite. For $\nabla^2 I$ to be semi-negative definite, there are two conditions to be satisfied [9]. The first condition is that its first order principle minor must be negative. As $\frac{\partial^2 I}{\partial p_i^2} = -\frac{S_i^2A_i^4}{S_1A_1p_1 + S_2A_2p_2 + \lambda} < 0$, this condition holds. The second condition is that the determinant of the Hessian matrix must be positive. It is easy to check that

$$|\nabla^2 I| = (A - B - C + D) - \frac{2S_1A_1S_2A_2}{S_1A_1p_1 + S_2A_2p_2 + \lambda} - (A - B - C + D),$$

(22)

The two terms on the right hand side can be dealt separately. First, we show that $A - B - C + D > 0$.

Lemma 4. $A - B - C + D > 0$.

Proof. The four terms in the statement are of the form $x \log(x)$. Let’s take $h(x) = x \log(x)$, then using properties of the function, we know that $h$ is a strictly convex function, and hence $h'$ is a strictly increasing function. Let $a = \lambda$, then using the mean value theorem, we have:

$$\exists x_1 \in (a,b) \text{ s.t. } h'(x_1) = \frac{h(b) - h(a)}{b - a},$$

$$\exists x_2 \in (c,d) \text{ s.t. } h'(x_2) = \frac{h(d) - h(c)}{d - c}.$$

Without loss of generality we can assume that $S_1A_1 < S_2A_2$, then we will have $a < b < c < d$. As $h'$ is an increasing function and $x_1 < x_2$, we have $h'(x_1) < h'(x_2)$ and $b - a = d - c$, then:

$$\frac{h(b) - h(a)}{b - a} < \frac{h(d) - h(c)}{d - c}$$

$$\frac{h(b) - h(a)}{b - a} < \frac{h(d) - h(c)}{d - c}.$$

Hence $h(d) + h(a) > h(b) + h(c)$. \(\square\)

As the first term is always greater than 0, for the function to be concave, the second term, $-\frac{2S_1A_1S_2A_2}{S_1A_1p_1 + S_2A_2p_2 + \lambda} - (A - B - C + D)$, must also be always positive. This, however, is not true. For example, taking $p_1 = 1, p_2 = 1$ and setting $S_1A_1 = 50, S_2A_2 = 100$ and $\lambda = 0.5$, the second term results in the value of $-26.4292$. Hence, we can conclude that $I_{X_1,X_2;Y}(p_1,p_2)$ is not always concave.

Appendix B
Proof of Lemma 2

In this Appendix, we calculate $\lim_{S_2A_2 \to \infty} g(0), \lim_{S_2A_2 \to \infty} g(1)$ and $\lim_{S_2A_2 \to \infty} f(0) = f(1)$.

$$\lim_{S_2A_2 \to \infty} g(0) = \lim_{S_2A_2 \to \infty} \left( \frac{1}{S_2A_2} \exp \left( \frac{1}{S_2A_2} (-A + C - S_2A_2) \right) + \frac{\lambda}{S_2A_2} \right)$$

$$= \lim_{S_2A_2 \to \infty} \left( \frac{1}{S_2A_2} \exp \left( \log(\lambda) \frac{S_2A_2 + \lambda}{S_2A_2} \right) + \log(S_2A_2 + \lambda) \frac{S_2A_2 + \lambda}{S_2A_2} \right)$$

$$= \lim_{S_2A_2 \to \infty} \left( \frac{1}{S_2A_2} \exp \left( \log(\lambda) \frac{S_2A_2 + \lambda}{S_2A_2} \right) \right)$$

$$= \lim_{S_2A_2 \to \infty} \left( \frac{1}{S_2A_2} \exp \left( \log(\lambda) \frac{S_2A_2 + \lambda}{S_2A_2} \right) \right) \left( S_2A_2 + \lambda \right) \frac{S_2A_2 + \lambda}{S_2A_2}$$

$$= \lim_{S_2A_2 \to \infty} \left( 1 + \frac{\lambda}{S_2A_2} \right)^{1 + \frac{\lambda}{S_2A_2}}$$

As

$$\lim_{S_2A_2 \to \infty} \frac{\lambda}{S_2A_2} = 1,$$

and

$$\lim_{S_2A_2 \to \infty} S_2A_2^{\frac{\lambda}{S_2A_2}} = \lim_{S_2A_2 \to \infty} \exp \left( \frac{\lambda}{S_2A_2} \log(S_2A_2) \right)$$

$$= \lim_{S_2A_2 \to \infty} \exp \left( \frac{\lambda}{S_2A_2} \log(S_2A_2) \right)$$

$$= 1,$$

and

$$\lim_{S_2A_2 \to \infty} \left( 1 + \frac{\lambda}{S_2A_2} \right)^{1 + \frac{\lambda}{S_2A_2}} = 1.$$

Hence, we obtain $\lim_{S_2A_2 \to \infty} g(0) = \frac{1}{e}$. 

\(\square\)
Similarly
\[
\lim_{S_2A_2 \to \infty} g(1) = \lim_{S_2A_2 \to \infty} \left( \frac{1}{S_2A_2} \exp \left( \frac{1}{S_2A_2} (-B + D - S_2A_2) \right) + S_1A_1 + \lambda \right) S_2A_2
\]
\[
= \lim_{S_2A_2 \to \infty} \left( \frac{1}{S_2A_2} \exp \left( \log(S_1A_1 + \lambda) \frac{-(S_1A_1 + S_2A_2 + \lambda)}{S_2A_2} \right) + \log(S_1A_1 + S_2A_2 + \lambda) \frac{(S_1A_1 + S_2A_2 + \lambda)}{S_2A_2} \right)
\]
\[
= \lim_{S_2A_2 \to \infty} \left( \frac{1}{S_2A_2} e^{(S_1A_1 + \lambda) \left( \frac{S_1A_1 + S_2A_2 + \lambda}{S_2A_2} \right)} \left( S_1A_1 + S_2A_2 + \lambda \right)^{\left( \frac{S_1A_1 + S_2A_2 + \lambda}{S_2A_2} \right)} \right)
\]
\[
= \lim_{S_2A_2 \to \infty} \left( \frac{1}{e} \left( S_1A_1 + \lambda \right)^{-\left( \frac{S_1A_1 + \lambda}{S_2A_2} \right)} \left( 1 + \frac{S_1A_1 + \lambda}{S_2A_2} \right)^{\left( \frac{S_1A_1 + \lambda}{S_2A_2} \right)} \right)
\]
\[
= \frac{1}{e}
\]

Now for the \(f(p_1)\), we notice that \(\lim_{S_2A_2 \to \infty} f(0) = \lim_{S_2A_2 \to \infty} f(1)\). Hence we calculate \(\lim_{S_2A_2 \to \infty} f(0)\).

\[
\lim_{S_2A_2 \to \infty} f(0)
= \lim_{S_2A_2 \to \infty} \frac{-B + A - \frac{S_1A_1}{S_2A_2} (C - A)}{A - B + C + D}
\]
\[
= \lim_{S_2A_2 \to \infty} \left( \frac{S_1A_1}{S_2A_2} \frac{\lambda \log (1 + \frac{S_1A_1}{\lambda}) + S_1A_1}{S_2A_2 + \lambda} + \frac{S_1A_1}{S_2A_2} \frac{S_2A_2}{S_2A_2 (S_2A_2 + \lambda)} \right)
\] 
\[
= \lim_{S_2A_2 \to \infty} \frac{-S_1A_1 \lambda \log (1 + \frac{S_1A_1}{\lambda}) + S_1A_1 S_2A_2}{2S_2A_2 (S_2A_2 + \lambda) \log (1 + \frac{S_1A_1}{S_2A_2 + \lambda}) + (S_2A_2)^2 \log (1 + \frac{S_1A_1}{S_2A_2 + \lambda}) + \frac{S_2A_2}{S_2A_2 (S_2A_2 + \lambda)} - \frac{S_2A_2 S_1A_1}{2(S_2A_2 + \lambda)}}
\]
\[
= \lim_{S_2A_2 \to \infty} \frac{\left( S_2A_2 + \lambda \right) \log \left( 1 + \frac{S_1A_1}{S_2A_2 + \lambda} \right) + S_2A_2 \log \left( 1 + \frac{S_1A_1}{S_2A_2 + \lambda} \right) - \frac{S_2A_2 S_1A_1}{2(S_2A_2 + \lambda)}}{S_2A_2 \log \left( 1 + \frac{S_1A_1}{S_2A_2 + \lambda} \right) + S_1A_1}
\]
\[
= 1,
\]

where (a) follows from the L’Hospital rule and (b) follows from multiplying by \(\frac{(S_2A_2)^2 (S_2A_2 + \lambda)}{(S_2A_2)^2 (S_2A_2 + \lambda)}\) and L’Hospital rule.

**APPENDIX C**

**PROOF OF LEMMA 3**

Using \(S_1A_1 = S_2A_2\), we will show that
\[g(1) - 1 < 0.\]