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On the Null-Spaces of Partially Elliptic Operators of a Certain Type

JAMES A. LAVITA

University of Denver, Denver, Colorado 80210

AND

HOMER F. WALKER

University of Houston, Houston, Texas 77004

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1. INTRODUCTION

Recently, various authors have studied classes of nonelliptic partial differential operators arising in mathematical physics, for example [2, 3, 5]. One can show, among other results, that a coerciveness inequality holds for operators of the form

$$A = \sum_{|\alpha| \leqslant m} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$$

(where $x \in \mathbb{R}^n$, each $a_\alpha(x)$ is a $k \times k$ matrix, and the notation is standard multi-index notation) provided the following conditions are satisfied: If the *symbol* of A is given by

$$a(x, \xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\, \xi^\alpha,$$

then

(i) $a(x, \xi)$ is of constant rank,

(ii) the null-space of $a(x, \xi)$ is independent of x.

Operators satisfying (i) and (ii) will be called *partially elliptic* operators. They are assumed to act on functions in the dense subset $H_m(\mathbb{R}^n; \mathbb{C}^k)$ of $L_2(\mathbb{R}^n; \mathbb{C}^k)$, denoted henceforth by H_m and L_2. The coerciveness of such operators holds on the set $N(a)^\perp \cap H_m$, where $N(a)^\perp$ is the orthogonal complement in L_2 of the set

$$N(a) = \{u: a(x, \xi)\, \hat{u}(\xi) = 0\}.$$

In scattering theory, it is important to know something about the total null-space $N(A)$ of such an operator A appearing in an evolution equation

$$u_t = Au.$$

Indeed, if A has no point spectrum other than zero, then it is to be expected that $N(A)$ is precisely the set on which scattering does not take place. Clearly $N(A) \supseteq N(a)$. To what extent can $N(A)$ exceed $N(a)$? In the following, we show that if A belongs to a certain class of first-order partially elliptic operators, then $N(A) = F_A \oplus N(a)$, where the dimension of F_A is finite and varies upper-semicontinuously with A in a certain sense. The development of these results closely parallels that of [6], using the coerciveness inequalities of [3]. Clearly, these results can be broadly extended in the spirit of [7] and [4]. However, in our opinion, energy would be better directed toward relaxing the condition (ii).

2. The Results

Let $A_0 = \sum_1^n a_j(\partial/\partial x_j)$ be a first-order partial differential operator with symbol $a_0(\xi) = \sum_1^n a_j\xi_j$. For each nonzero $\xi \in \mathbb{R}^n$, let $p(\xi)$ be the matrix of the projection onto the null-space of $a_0(\xi)$, and let $U(\xi)$ be a unitary matrix mapping the null-space of $a_0(\xi)$ onto the orthogonal complement of the range of $a_0(\xi)$. (We may assume that $U(\xi)$, as well as $p(\xi)$, is homogeneous of degree zero in ξ.) Then, for each $\xi \neq 0$,

$$b_0(\xi) = a_0(\xi) + |\xi| U(\xi) p(\xi)$$

is a nonsingular matrix, and the pseudo-differential operator B_0 with symbol $b_0(\xi)$ is an elliptic operator on L_2 with domain H_1. Define $M(B_0, R) = \{u \in H_1: B_0u(x) = 0 \text{ for } |x| \geqslant R\}$. We establish the following fundamental property of this set.

Lemma 1. *Subsets of $M(B_0, R)$ which are bounded in H_1 are precompact in L_2.*

Proof. The proof is, in essence, the Fourier-transform proof of the Rellich Compactness Theorem (see [1]).

Suppose $\{u_n\} \subseteq M(B_0, R)$ is a bounded sequence in H_1. Set $f_n = B_0u_n$. Since $\{f_n\}$ is an L_2-bounded sequence of functions with support in $\{x \in \mathbb{R}^n: |x| \leqslant R\}$, the Fourier transforms $\hat{f}_n(\xi)$ are uniformly bounded with uniformly bounded derivatives in $\mathbb{R}^n$. It follows from the Arzela–Ascoli Theorem that there exists a subsequence $\{\hat{f}_{n_i}\}$ which converges uniformly on every compact subset of $\mathbb{R}^n$.

Now for each $\rho > 0$, one has

$$\| u_{n_j} - u_{n_k} \|^2 = \int_{|\xi| \geqslant \rho} | b_0(\xi)^{-1} [\hat{f}_{n_j}(\xi) - \hat{f}_{n_k}(\xi)] |^2 \, d\xi$$

$$+ \int_{|\xi| \leqslant \rho} | b_0(\xi)^{-1} [\hat{f}_{n_j}(\xi) - \hat{f}_{n_k}(\xi)] |^2 \, d\xi$$

$$\leqslant \frac{C}{\rho^2} \| \hat{f}_{n_j} - \hat{f}_{n_k} \|^2 + C \int_{|\xi| \leqslant \rho} \frac{1}{|\xi|^2} | \hat{f}_{n_j}(\xi) - \hat{f}_{n_k}(\xi) |^2 \, d\xi$$

where $C = \sup_{|\xi|=1} | b_0(\xi)^{-1} |^2$. Since the sequence $\{f_{n_j}\}$ is bounded in L_2, the first term on the right may be made small by taking ρ large. If $n \geqslant 3$, then $\int_{|\xi| \leqslant \rho} (1/|\xi|^2) \, d\xi < \infty$, and the uniform convergence of the functions $\hat{f}_{n_j}$ implies that the second term on the right approaches zero as $j, k \to \infty$. If $n = 2$, then $\hat{f}_{n_j}(0) = 0$ for each j, since $\hat{u}_{n_j}(\xi) = b_0(\xi)^{-1} \hat{f}_{n_j}(\xi)$ is in L_2. Since the functions $\hat{f}_{n_j}$ have uniformly bounded derivatives, the second term of the right must approach zero as $j, k \to \infty$ when $n = 2$. It follows that $\{u_{n_j}\}$ is Cauchy in L_2 for all $n \geqslant 2$, and the proof is complete.

We now define $PE(A_0, R)$ to be the set of partial differential operators of the form

$$A = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} + b(x)$$

which are such that

(i) $a_j(x) = a_j$ and $b(x) = 0$ for $|x| \geqslant R$,

(ii) for each x and ξ in $\mathbb{R}^n$, the null-space of $a(x, \xi) = \sum_{j=1}^{n} a_j(x) \xi_j + b(x)$ is the same as the null-space of $a_0(\xi)$.

If A is an operator in $PE(A_0, R)$ then the pseudodifferential operator B, with symbol $b(x, \xi) = a(x, \xi) + |\xi| U(\xi) p(\xi)$ is elliptic. Then a standard coerciveness inequality

$$\| u \|_1 \leqslant C[\| u \| + \| Bu \|] \tag{1}$$

holds for B (see, for example, [3]). Since $N(B) \subseteq M(B_0, R)$, it follows from this inequality and from Lemma 1 that $\{u \in N(B) : \| u \| = 1\}$ is compact. Hence $N(B)$ is finite-dimensional.

Theorem 1. *If A is in $PE(A_0, R)$, then*

$$N(A) = N(a_0) \oplus F_A$$

where F_A is finite-dimensional and $N(a_0) = \{u \in L_2 : a_0(\xi) \hat{u}(\xi) = 0\}$.

Proof. Since $N(a_0) \subseteq N(A)$, one has

$$N(A) = (N(A) \cap N(a_0)^\perp) \oplus (N(A) \cap N(a_0))$$
$$= F_A \oplus N(a_0)$$

where $F_A = N(A) \cap N(a_0)^\perp$. Since $F_A \subseteq N(B)$, F_A must be finite-dimensional.

It will now be shown that the dimension of F_A depends upper-semi-continuously on an operator A in $PE(A_0, R)$.

LEMMA 2. *If B is the pseudodifferential operator associated with an operator A in $PE(A_0, R)$, then there exists a $C > 0$ for which*

$$\| Bu \| \geqslant C \| u \|$$

for u in $M(B_0, R) \cap N(B)^\perp$.

Proof. Suppose there exists no such constant. Then one can find a sequence $\{u_n\} \subseteq M(B_0, R) \cap N(B)^\perp$ such that $\| u_n \| = 1$ and $\| Bu_n \| \to 0$. The inequality (1) implies that $\{u_n\}$ is bounded in H_1, and so, by Lemma 1, there is a subsequence $\{u_{n_j}\}$ which is Cauchy in L_2. Since B is a closed operator, $\lim_{j\to\infty} u_{n_j} \equiv u_0$ must be in $N(B)$. Since u_0 is the limit of functions in $N(B)^\perp$ having norm 1, this is a contradiction.

Now if A and $\tilde{A}$ are two operators in $PE(A_0, R)$, it follows from the boundedness of the coefficients of these operators that there exists an $\epsilon > 0$ for which

$$\|(A - \tilde{A})\, u \| \leqslant \epsilon \| u \|_1 \tag{2}$$

for all u in H_1. In particular, ϵ may be taken small if the coefficients of $\tilde{A}$ are near those of A uniformly in $\mathbb{R}^n$.

THEOREM 2. *If A is an operator in $PE(A_0, R)$, then there exists an $\epsilon_A > 0$ such that*

$$\text{dimension}\, F_{\tilde{A}} \leqslant \text{dimension}\, F_A$$

for every operator $\tilde{A}$ in $PE(A_0, R)$ satisfying $\|(A - \tilde{A})\, u \| \leqslant \epsilon_A \| u \|_1$ for all $u \in H_1$.

Proof. Given A and $\tilde{A}$ in $PE(A_0, R)$, let B and $\tilde{B}$ be the associated pseudodifferential operators. Then by Lemma 2 and the inequalities (1) and (2), there exist constants C and ϵ such that

$$\| u \|_1 \leqslant C \| Bu \| \leqslant C \| \tilde{B}u \| + C \|(B - \tilde{B})\, u \|$$
$$= C \| \tilde{B}u \| + C \|(A - \tilde{A})\, u \|$$
$$\leqslant C \| \tilde{B}u \| + C\epsilon \| u \|_1$$

for all $u \in M(B_0, R) \cap N(B)^{\perp}$. If $\epsilon \leqslant \epsilon_A < 1/C$, then there exists a different constant C, independent of ϵ, for which

$$\| u \|_1 \leqslant C \| \tilde{B} u \|$$

for all $u \in M(B_0, R) \cap N(B)^{\perp}$.

Suppose $\tilde{A}$ is an operator in $PE(A_0, R)$ with $\epsilon \leqslant \epsilon_A$ as above, and suppose that $\dim F_{\tilde{A}} > \dim F_A$. Then there exists a nonzero u_0 in $F_{\tilde{A}} \cap F_A{}^{\perp}$. It follows that $u_0 \in M(B_0, R) \cap N(B)^{\perp}$, and so $\| u_0 \|_1 \leqslant C \| \tilde{B} u_0 \| = 0$, which is a contradiction.

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