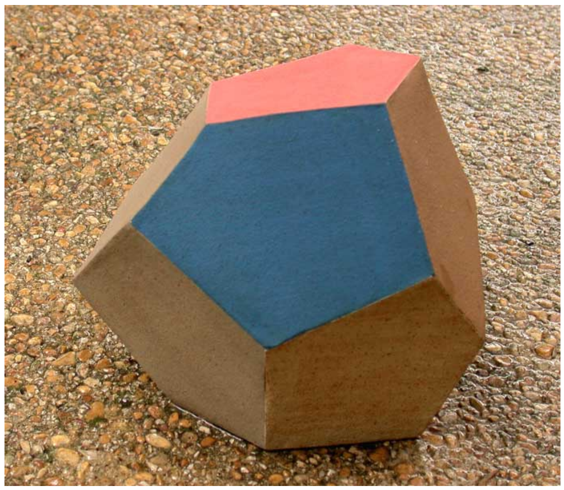
The SIMPLEX METHOD is an iterative method for solving linear programming problems. We discuss the method from both geometric and algebraic vantage points.

The Simplex Method reduces things to solving equivalent systems of linear equations.

We start with an LP in standard form

$$\begin{array}{ll} \text{maximize} & z = \mathbf{c}^T \mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b}, \ \ \mathbf{x} \ge 0. \end{array}$$



The bounds (faces) of the polytope come from the n equalities $x_i = 0, i = 1, \cdots, n$ (the original decision variables), and also from the m equalities found in the matrix equation $Ax = b$:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ &\vdots \\ x_n &= 0 \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Each of these linear equations gives a hyperplane. (In 2-space, a linear equation in x, y gives a line. In 3-space, a linear equation in x, y, z gives a plane that we can draw.)

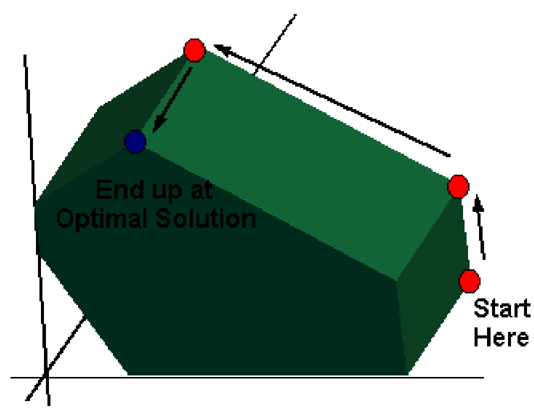
When we are in n-space, the intersection of n hyperplanes, generally speaking, gives us a point; the intersection of $n-1$ hyperplanes gives us an “edge”.

Candidates for the optimal solution of an LP are the corner points of the polytope in n-space and they correspond to the intersection of n hyperplanes.

FACT: If an LP has an optimal solution, then if a corner point feasible (CPF) solution has no adjacent CPF solution that gives a better objective value then the current CPF solution is optimal.

Simplex Method Geometrically:

- **Initialization** Start with a corner point on the polytope.
- **Optimality Test** Will the value of z increase if we move to an adjacent CPF?
  - If no, then this is the optimal solution $x^\star$ so STOP.
  - If yes, then continue.
- **Determine the Direction of Movement** Consider the edges along which the value of z increases. Along which edge will z increase most rapidly?
- **Determine where to Stop** Slide along this edge until we reach a new hyperplane - equality of one of the constraints. If we go any further we will violate a constraint. We are now at a new CPF.
- Go back up to **Optimality Test**.



Simplex Method Algebraically:

Introduce **slack variables** $\mathbf{w} = (x_{n+1}, x_{n+2}, \cdots, x_{n+m})$ which take on the value of the space between $A\mathbf{x}$ and $\mathbf{b}$. We define $\mathbf{w} = \mathbf{b} - A\mathbf{x}$. We then write the **augmented form of the LP**:

$$\begin{array}{ll} \text{maximize} & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} + \mathbf{w} = \mathbf{b}, \;\; \mathbf{x}, \mathbf{w} \geq 0. \end{array}$$

FACT: Any solution $(x_1, \cdots, x_{n+m})$ to the system of equations

$$A\mathbf{x} + \mathbf{w} = \mathbf{b} \tag{1}$$

with $x_i \geq 0$ is a feasible solution to the original LP.
An **augmented solution** is a solution for the original variables that has been augmented by the corresponding values of the slack variables. That is, it is a solution of (1).

We can see that the faces/hyperplanes of the polytope come from the $n + m$ equations $x_1 = 0, x_2 = 0, \cdots, x_n = 0, x_{n+1} = 0, \cdots, x_{n+m} = 0$. That is, every face (hyperplane) of our feasible region is associated with one of the $n + m$ variables being equal to zero.

The fact that every corner of the feasible region is associated with the intersection of n hyperplanes, i.e. simultaneous n variables are equal to 0, leads us to the realization that we need to look at solutions to $A\mathbf{x} + \mathbf{w} = \mathbf{b}$ where n of the $n + m$ variables are zero and the others are ≥ 0. These are called **basic feasible solutions** (BFS).

When you think of a corner point feasible (CPF) point, think simultaneously of the basic feasible solution (BFS). They are the same except the CPF just reports the value of the original variables $x_1, \cdots, x_n$ whereas the BFS reports the slack values, too.

Of the $n + m$ variables of a BFS, m of them are designated as **basic** variables and the other n are **non-basic** variables.

The n non-basic variables are set to zero. Then the system $A\mathbf{x} + \mathbf{w} = \mathbf{b}$ is solved to find the values of the remaining m basic variables.

There are $n + m$ variables and only m equations in (1) so when we force n of them to be zero, we are left to solve the m equations for m unknowns which, generally but not always, gives us a single solution.
As long as these basic variables that we solve for are ≥ 0, this is a BFS and it is associated with a corner point on the feasible region.