WORCESTER POLYTECHNIC INSTITUTE

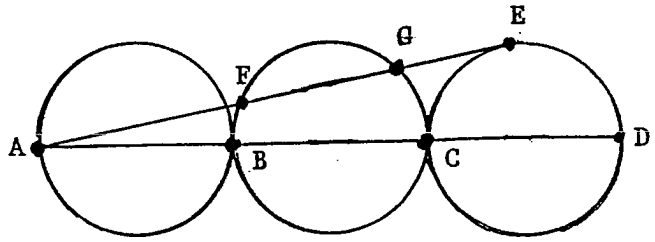
ELEVENTH ANNUAL INVITATIONAL MATH MEET

OCTOBER 22, 1998

TEAM EXAM QUESTION SHEET

DIRECTIONS: Please write your answers on the Team Answer Sheet provided. This part of the contest is 30 minutes. Each correct answer to questions 1-14 is worth 3 points. Calculators MAY NOT be used.

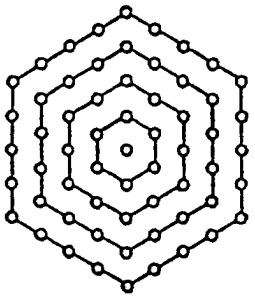
1 The points A,B,C and D are collinear where AB, BC and CD are the diameters of three circles, each of radius 10. Let the line AE be tangent to the circle through C and D at the point E, as shown in the diagram. The line AE intersects the circle through B and C at the points F and G as shown. Find the length of the chord FG.



2 Given that $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}=2$, simplify the expression $\frac{a^2}{b+c}+\frac{b^2}{a+c}+\frac{c^2}{a+b}$.

3 Let $f(x)=3+|x-4|+2|x-6|$, $2\le x\le 8$. Find the area of the planar region determined by the graphs of $y=f(x)$, $y=1$, $x=2$ and $x=8$.

[4] Consider hexagonal array shown at the right. The nth hex number, h_n, is defined as the total number of dots in the first n layers. The first seven hex numbers are 1, 7, 19, 37, 61, 91, 127. Find h_{50}, the 50th hex number.



[5] Let $g(x) = \sqrt{x + 2\sqrt{x-1}} - \sqrt{x - 2\sqrt{x-1}}$, $x \geq 1$. Compute $\sum_{k=1}^{1000} g(k)$.

[6] An ant moves across the top of a square at 4 mph, down the right side at 3 mph, across the bottom at 2 mph, and up the left side at 1 mph. What is the average speed of the ant, in mph, for one complete trip around the square?

[7] How many odd numbers between 1000 and 9999 have distinct digits?

[8] Suppose that at the end of any year, a unit of money has lost 20% of the value it had at the beginning of that year. Find the least integer n such that at the end of n years the unit of money will have lost at least 80% of its initial value. (Assume $\log_{10} 2 = 0.301$.)

[9] Twin primes are prime numbers which differ by 2 (17 and 19, or 29 and 31, for example). There are many twin primes, but there is only one triplet of primes: 3, 5 and 7. Which of the following best express the reason(s) why there cannot be any other triplet of primes?

[a] One of the three integers would be a perfect square (in the sequence 23, 25, 27, for example, 25 is a perfect square),

[b] In any sequence of three integers, if successive integers differ by 2, then one of the three integers is divisible by 3.

[c] If one divides the first integer in such a sequence by the last, the result is less than 1.

[d] $3^2 + 5^2 < 7^2$ while this is not true for large integers. For example $(23)^2 + (25)^2 > (27)^2$.

10 What is the coefficient of a^4b^2c in the expansion of $(a+2b+c)^7$?

11 Find the number of lines in a three-dimensional rectangular coordinate system that pass through five distinct points of the form (i, j, k), where i, j, k are positive integers not exceeding 5.

12 In the sequence of numbers 1, 4, 3 ... each term after the first two is equal to the term preceding it minus the term preceding that. Find the sum of the first one hundred terms of the sequence.

13 Compute the value of $\left(1-\frac{3}{7}\right)\left(1-\frac{3}{8}\right)\left(1-\frac{3}{9}\right)\left(1-\frac{3}{10}\right)\cdots\left(1-\frac{3}{20}\right)$.

14 Let z be a complex number of magnitude 1, $z \neq -1$. Let $i = \sqrt{-1}$, and let c and d be real numbers with $c + di = \frac{1}{1+z}$. Find c.